Waiting for Capital: Dynamic Intermediation in Illiquid Markets*

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Abstract

We consider a firm that has an internal carry cost of cash and infrequent access to public capital markets. In the interim, it contracts with a deep-pocketed but costly intermediary subject to limited commitment. Under the optimal financing agreement, the intermediary absorbs a fraction of cash-flow risk that decreases in the firm's net-cash position. A credit-line with a state-dependent interest rate and early-repayment incentives, restricted equity that vests upon market access, and cash reserves jointly implement the optimal contract. Initially, the firm uses cash and the credit-line simultaneously. If the firm runs out of cash, it either liquidates or uses restricted-equity and the credit-line to sustain operations. Thus, an overlapping pecking order arises. Upon market access, the firm issues common equity to retire the credit line. Although the intermediary has deep pockets, it allows riskier firms to face endogenous liquidation. To ensure the survival of less risky firms, it acquires restricted-equity that it sells upon market access. The set of firms facing liquidation is smaller if market access, and therefore intermediary exit, is more likely. Importantly, the optimal financing agreement is renegotiation proof.

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The assumption of frictionless financial markets, while plausible over the long run for most firms, is often at odds with the reality facing many, especially smaller, firms: raising capital at short-notice usually is costly and/or constrained, while immediate access to frictionless financial markets is unavailable. Such short-notice financing is often provided by specialized intermediaries that are under-diversified and/or have a higher cost of funds, e.g., due to regulatory constraints. Meanwhile, access to frictionless markets takes time: asymmetric or hard-to-verify information, as well as some soft information, might only be accessible after some time investment by participants in the frictionless market, while it is immediately available to specialized intermediaries. Consequently, we observe that contracts with such intermediaries often result in more complex arrangements than the frictionless financial instruments we see for larger firms: credit lines with performance-sensitive interest rates, flexible equity stakes, convertible debt and so forth. Further, exits from such contracts occur upon broader market access, which often feature pay-downs of accumulated credit lines and sales of built-up equity stakes by the intermediary.

We construct a tractable continuous-time model to capture the salient features of the disparity between short-notice financing and frictionless financial markets. A firm owned by risk-neutral outside shareholders is subject to risky cash flows and faces financial constraints. Specifically, the firm can only issue equity without cost to new risk-neutral investors at Poisson times, and must thus "wait for capital." In the interim, the firm must finance operating losses with short-notice financing. It can either provide itself with such financing with internal cash reserves or contract with an ever-present intermediary. Both liquidity facilities are costly: cash held in the firm earns a return below the risk-free rate due to an internal carry cost of cash, while financing from the intermediary must compensate her for bearing risk. Specifically, we endow the intermediary with risk-averse CARA preferences as a proxy for its own financial constraints. We derive the optimal contract between the firm and the intermediary.

We first show that we can summarize the state of the firm with a single variable that we term excess cash. This variable equals the firm's cash-holdings less the intermediary's deferred payouts measured in dollars. We then derive the optimal financing contract by maximizing the outside shareholder's value using the dynamic programming approach. This approach boils down to solving an ordinary differential equation (ODE) over excess cash with two free boundaries. At the upper boundary of excess cash, the firm pays a dividend to outside shareholders. At the lower boundary, the intermediary either liquidates the firm or temporarily assumes all cash flow risk. We show that the latter is only possible for firms with high enough first-best net present value (NPV) compared to their risk. Thus, firms with greater cash-flow risk face potential liquidation, while the intermediary rescues firms with smaller cash-flow risk in exchange for a restricted equity stake. Notably, the willingness of the intermediary to temporarily absorb all cash-flow risk increases with financial market access, i.e., the probability of being able to issue new outside equity, as it provides an avenue for the intermediary to cash out her equity stake in the firm.

We show that the firm uses cash and intermediary financing to absorb cash-flow shocks, effectively saving and borrowing, at the same time. For example, the firm may cover a $1 operating loss by withdrawing $.35 from its internal cash and raising $.65 in financing from the intermediary. Symmetrically, if the firm makes a $1 operating profit, it may deposit $.35 in its cash account and repay $.65 to the intermediary. The fraction of risk covered by cash and the intermediary depends on the state of the firm. The outside shareholders are effectively risk-averse with respect to cash-flow risk because of the firm's financial constraints. Moreover, the shareholders become more risk-averse as the firm depletes its internal cash. At the same time, the intermediary demands a constant compensation for bearing cash-flow risk. As such, the shareholders and the intermediary face a state-dependent risk sharing problem. The optimal financing agreement thus calls for the intermediary to bear an increasing fraction of cash flow risk as the firm depletes its cash.

As long as the firm maintains a positive cash balance, the optimal contract does not defer payments to the intermediary. If the firm runs out of cash, and the intermediary is willing to keep it alive, then the contract grants the intermediary a deferred payout that is repaid at the first time the firm has access to new outside shareholders. This arrangement effectively shifts payouts to the intermediary from states in which the firm is constrained, i.e., out of cash, to states in which the firm is unconstrained. However, this feature is subject to a limited commitment problem for the outside shareholders: they can always refuse to raise new capital. We show this constraint binds at the optimum and therefore reduces the set of firms that the intermediary will continue to fund if they run out of cash.

We show the optimal contract can be implemented via a credit line that features a state-dependent interest rate and early repayment incentives upon refinancing to overcome limited commitment, and restricted equity grants that vest upon refinancing, with an overlapping pecking-order: even though cash and debt are used simultaneously, as cash dwindles the firm relies more on debt, until such a time when the firm is out of cash, at which point it switches to debt and (internal) equity if it does not liquidate. This internal equity can be thought of as a form of private equity financing. One example of this type of financing is private investment in public equity (PIPE) of distressed firms.

Because the optimal policies are Markov, refinancing is optimally to the dividend payout boundary, and deferred payouts are only used when the firm holds no more cash, the contract is renegotiation proof. Renegotiation proofness allows a broader interpretation of the intermediary, as it effectively allows the intermediary to “switch” in the future, say after refinancing. Any anticipation of such a ”switch”, and consequent renegotiation, does not matter for any current contract by the renegotiation proofness result.

Our paper mainly relates to the literature on dynamic corporate liquidity management, pioneered by Bolton, Chen, and Wang (2011) and Décamps, Mariotti, Rochet, and Villeneuve (2011). In a unified model of corporate investment, financing, and liquidity management, Bolton et al. (2011) demonstrate how liquidity management and firm financing interacts with a firm's investment decisions. Further contributions in this literature include Gryglewicz (2011), Rochet and Villeneuve (2011), Bolton, Chen, and Wang (2013), Hugonnier, Malamud, and Morellec (2015); Hugonnier and Morellec (2017), Malamud and Zucchi (2018), and, more recently, Abel and Panageas (2020b,a) and Bolton, Li, Wang, and Yang (2021). In addition, Bolton, Wang, and Yang (2021) study dynamic liquidity management with short-term debt financing, thereby highlighting the interaction between the endogenous pricing of the debt and the optimal the equity payout and issuance strategies. Our paper differs from these papers, as it adds an intermediary who can provide any type of financing to a dynamic liquidity management model.

There is also an extensive literature of applying structural models to capture the optimal liquidity policies of a firm subject to a range of financial market frictions with key contributions including Hennessy and Whited (2005, 2007); Hennessy, Levy, and Whited (2007), and Whited and Wu (2006). In a recent paper, Nikolov, Schmid, and Steri (2019) introduce a separate credit line with exogenous hedging properties that differentiate it from cash, so that a firm uses cash and the credit-line simultaneously as in our model. Our paper derives some of properties of a credit line that distinguish it from cash in an optimal contracting framework.

Our model features a limited commitment problem and is thus related to the literature that explores how such a contraint restricts the contracting space as first shown in Hart and Moore (1994). Specifically we relate to the dynamic contracting literature that studies optimal risk-sharing between a principal and an agent under limited commitment, such as Ai and Li (2015) and Bolton, Wang, and Yang (2019). The closest reference to our paper is Bolton et al. (2019). Our model differs from theirs in that we allow the deep-pocketed but risk-averse intermediary to provide the marginal financing of the firm. Or model is complementary to theirs in that it highlights optimal financing from an under-diversified intermediary in a world of physical cash constraints and limited commitment to refinance from the shareholders, whereas their model is driven by the connection between investment, scale of the firm and the scale of the manager's outside option in an otherwise complete market world. More broadly related, DeMarzo and He (2021) build a model in which the key friction is the inability of equity holders to commit to a future debt issuance strategies. Our model concerns itself with a related situation, the inability of equity holders to commit to a future share issuance policy. Effectively, share holders cannot commit to diluting themselves to their own detriment, restricting the contracting space. Rampini and Viswanathan (2010) and Rampini, Sufi, and Viswanathan (2014) provide models in which the optimal state- and time-varying allocation of net-worth in a complete market is restricted by limited commitment in the form of limited enforcement which implies a role for net worth in easy financial constraints. In their work, risk-management requires net worth that could otherwise be used for productive investment.

There is a large empirical literature on the connection between cash, credit lines, and firms' policies. Early work by Smith and Stulz (1985) and Opler, Pinkowitz, Stulz, and Williamson (1999) provides overviews on the determinants of firms' hedging and cash holding polices. Berger and Udell (1995) show that the length of a banking relationship plays a key role in the level of credit lines provided to a small firm, reflecting information asymmetries. We interpret a key assumption in our model — the speed at which a firm can raise outside funding and the ability of the intermediary to provide immediate funding — as reflecting both the speed with which such information asymmetries can be overcome by the outside investors, and the privileged position of the intermediary as an informed relationship lender. Almeida and Campello (2007) and Campello, Graham, and Harvey (2010), amongst many, provide evidence how financial constraints affect corporate investment. In a theoretical setting, Acharya, Almeida, and Campello (2007) show that debt and cash are not perfect substitutes due to different hedging properties, and thus may be used simultaneously. They provide empirical evidence supporting this conclusion. Sufi (2009) provides further empirical evidence on the difference between the use of cash and lines of credit for firms. Other related empirical work related to cash holdings and financing choices are Leary and Roberts (2005), Bates, Kahle, and Stulz (2009), Eisfeldt and Muir (2016), amongst many others.

Finally, the intermediary in our model provides financing both in the form of a credit line and in the form of equity shares, resembling a private equity (PE) arrangement. The seminal paper on contracting within a venture capital framework is Kaplan and Stromberg (2001). More recently, Ewens, Gorbenko, and Korteweg (2021) provides a more in depth look at the prevailing contracts in the venture capital and private equity space.

1 Model Setup

We consider a firm whose assets under management produce volatile cash flows and is owned by dispersed risk-neutral *outside investors*, also referred to as outside equity holders. Access to outside capital is limited, but an *intermediary* is available to supply capital continuously at an elevated cost, with the cost scaling with the riskiness of the financing provided. To optimally receive financing from the intermediary, a payout agreement with limited commitment as well as limited liability is set up at time zero that maximizes firm value. We interpret the intermediary–outsiders relationship broadly. For instance, the intermediary can be interpreted as the "house bank" that has no information asymmetry vis-a-vis the firm, while other possible source of capital face information barriers that prevent them from quickly providing capital.

1.1 Cash Flows, Cash Holdings, and Transfers

Cash Flows. Let time t be continuous on $[0, \infty)$, and Z_t be a standard Brownian Motion. The firm has assets in place that generate cash flows dX_t with mean μ and volatility σ, i.e.,

$$dX_t = \mu dt + \sigma dZ_t. \tag{1}$$

Transfer Agreement. The transfer agreement, or contract, $\mathcal{C} = (Div, I, \Delta M)$ stipulates cumulative dividend payouts Div_t to outside investors, money raised upon financial market access ΔM_t, and cumulative transfers I_t to $(dI > 0)$ and from $(dI < 0)$ the intermediary.[1]

Financial market friction. We assume that it is not possible to raise external funds from outside investors except at Poisson times $d\Pi_t = 1$ that arrive with constant intensity π, with $d\Pi_0 = 1$ to reflect market access at the founding of the firm. We will refer to such capital market access as refinancing opportunity. Notation wise, we write $dDiv_t \geq 0$ to denote the financing constraint, and denote capital infusions by outsiders upon capital market access by $\Delta M_t d\Pi_t \geq 0$. We interpret the time it takes to arrange for financing as caused by (un-modelled) asymmetric information –

[1] Here, we are separating payments *to* outside shareholders, Div_t, from payments *from* outside shareholders at refinancing, ΔM_t.

the outside markets take time to verify information, while the intermediary, being a specialist, can circumvent this time delay.

Corporate Liquidity. The firm's financial constraints together with the fact that cash-flow shocks can be negative imply that the firm may have an incentive to build a buffer stock of cash via retained earnings. Specifically, as the outside investors are unable to inject cash, all cash-flow realizations dX_t, dividends $dDiv_t \geq 0$, and transfers to and from the intermediary dI_t flow through the firm's internal cash balance M_t. We normalize the cash balance at $t = 0_-$ to zero (i.e., $M_{0_-} = 0$). In contrast to outside investors, the intermediary can inject cash into the firm, $dI_t < 0$, but this source of financing will turn out to be costly and possibly limited due to the intermediary's elevated cost of funds. The cash balance held within the firm accrues interest at the rate $r - \lambda$ where r is the common interest rate and $\lambda \in (0, r)$ represents a *carry cost of cash*.[2] The dynamics of cash reserves M_t are then given by

$$dM_t = dX_t + (r - \lambda) M_t dt - dDiv_t - dI_t + \Delta M_t d\Pi_t. \tag{2}$$

The key assumption is that cash holdings are not allowed to turn negative, $M_t \geq 0, \forall t \geq 0$. This implies that if M_t attains zero, the intermediary must either inject the necessary funds or the firm must liquidate. Liquidation thus occurs at a stopping time $\tau \in [0, \infty]$, and $dDiv_t = dI_t = dX_t = 0$ for all $t > \tau$. For tractability, we assume that the liquidation value of the firm beyond its current cash-holdings M is zero.[3]

Transfer Process. The dynamics of the intermediary's cumulative transfers I_t are

$$dI_t = \mu_I(t) dt + \sigma_I(t) dZ_t + \alpha_I(t) d\Pi_t + \xi_I(t) dDiv_t. \tag{3}$$

We will solve for the process I_t as a residual implied by the optimal contract.

1.2 Preferences and Savings

A common discount rate $r > 0$ applies to all agents, and is equal to the risk-free rate.

[2]Impatient equity investors leads to similar results, but we follow Décamps et al. (2011) with a carry cost of cash.

[3]That is, the assets in place stop producing cash-flows upon liquidation, while the cash-balance is divvied up according to the contract.

Preferences. The firm's founder as well as its outside investors are risk neutral, reflecting that outside investors are diversified. To generate an elevated cost of funds, we assume that the intermediary is risk averse with CARA preferences with risk-aversion of $\rho > 0$, so that its instantaneous utility of consumption c is

$$u(c) = -\frac{1}{\rho}\exp(-\rho c). \tag{4}$$

Risk aversion's main role is to make funding from the intermediary expensive, the more so the riskier the funding. It either reflects the notion that the type of intermediaries we consider in this paper typically have undiversified exposure to the firm, or that the intermediary is subject to risk-based shadow costs of capital. Importantly, the intermediary has *limited commitment*. Thus, the intermediary's continuation value from following the payout agreement must at any time $t \geq 0$ exceed the intermediary's outside option, which we normalize to zero.[4] Similarly, the outside equity holders also have limited commitment (LC) and a zero outside option.[5] Effectively, LC implies that shareholders cannot commit to undertake actions that lower their continuation value, i.e., they cannot be forced to vote against their self interest at time of refinancing.

Intermediary's Savings account. The intermediary can maintain savings on its own account, denoted by S_t. Savings accrue interest at rate r and are subject to changes induced by transfers to/from the firm dI_t and consumption c_t, so that

$$dS_t = rS_t dt + dI_t - c_t dt. \tag{5}$$

Endowing the intermediary with the possibility to accumulate savings ensures that it can smooth its consumption beyond liquidation of the firm. Consumption c_t and savings balance S_t can both take positive and negative values. We normalize the balance of the savings account at $t = 0_-$ to zero (i.e., $S_{0_-} = 0$). Finally, savings must satisfy the standard transversality condition $\lim_{t\to\infty} \mathbb{E}\left[e^{-rt}S_t\right] = 0$, ruling out Ponzi schemes. For simplicity, the intermediary's savings are not subject to the carry cost of cash $\lambda > 0$.[6]

[4] Ai and Li (2015) and Bolton et al. (2011) present models in which the time-varying outside option plays a key role in determining the contract. We highlight a complementary mechanism by assuming a constant outside option.

[5] Note that our abbreviation LC refers to limited commitment by the shareholders only, as distinct from limited commitment by the intermediary.

[6] Our results remain largely unchanged if the intermediary's save at interest rate $r - \lambda$ rather than r.

1.3 Optimal security design problem

Suppose that the company's penniless founder designs an optimal contract with outside investors and the intermediary. The contract $\mathcal{C}$ can be written conditional on all publicly observable variables and outcomes. Given $\mathcal{C}$, the intermediary chooses consumption c_t to maximize its lifetime utility, with optimal consumption denoted c_t^*. Let $U(\mathcal{C})$ be the intermediary's indirect utility for a given contract $\mathcal{C}$, i.e.,

$$U(\mathcal{C}) = \max_{c_t} \mathbb{E}\left[\int_0^\infty e^{-rt} u(c_t)\, dt\right]. \tag{6}$$

We call the transfer agreement incentive compatible (IC) if it respects the intermediary's as well as the shareholders' limited commitment, and we restrict our attention to IC transfer agreements.

Then, consider an optimal contract $\mathcal{C} = (I, Div, \Delta M)$ with the restriction that it fulfills promise-keeping. The optimal financing contract that fulfills incentive compatibility solves

$$U_0 = \max_{\mathcal{C} \text{ is } IC} U(\mathcal{C}) = \max_{\mathcal{C} \text{ is } IC} \mathbb{E}_0\left[\int_0^\infty e^{-rt} u(c_t^*)\, dt\right]. \tag{7}$$

Note that because the outside investors are competitive and have zero outside option, their initial cash contribution ΔM_0 must equal the present value of the stream of dividends they expect to receive. We can interpret this quantity as the competitive price P_0 of the security (i.e., outside equity) that the intermediary issues and write

$$P_0 = \mathbb{E}\left[\int_0^\tau e^{-rt} dDiv_t\right] = \Delta M_0. \tag{8}$$

As $M_{0_-} = S_{0_-} = 0$, the initial cash balance of the firm M_0 is given by

$$M_0 = \underbrace{P_0}_{\text{Cash from outsiders}} + \underbrace{(-S_0)}_{\text{Cash from intermediary}} - \underbrace{V_0}_{\text{Cash extracted by founder}}. \tag{9}$$

1.4 Discussion of assumptions

Let us discuss two key assumptions we make that deviate from the literature and contrast them with alternative choices.

Refinancing constraint via Poisson times. As opposed to Décamps et al. (2011) and Bolton et al. (2011), who have fixed and variable cost of equity issuance, we follow Hugonnier et al. (2015) in assuming that the firm is unable to raise capital outside Poisson times. First, note that our assumption can be interpreted as a Markov chain variant in which the costs are infinite except for very short periods at which the costs vanish. The key difference, however, lies with how these assumptions link to possible firm liquidation: Under a costly equity issuance assumption, survival of the firm is driven by the fixed and variable costs parameters. Under our Poisson assumption, we show that survival of the firm is driven by the willingness of the intermediary to take an equity stake in the firm to bridge the time between market access. Thus, our assumptions are more relevant for the purposes of this paper. We relax the assumption of market access with a fixed Poisson intensity by introducing active search in Section 4.3.

Costly intermediary financing via CARA. We want to capture the reality that specialized intermediary financing is costly, the costlier the riskier. To capture this mechanism, we could have introduced an intermediary that is (i) itself cash constrained or (ii) has regulatory cost-of-capital that are linked to the riskiness of its financial agreements. Note that modeling it via (i) would result in higher cost of funds for riskier financing arrangements, as higher risk makes the intermediary hitting its own funding constraint more likely. But note that it also introduces a further state-variable, the cash-holdings of the intermediary. Modeling it via (ii) would require us to take a stand on how the riskiness of financing translates into regulatory cost of funds. We view CARA utility as a parsimonious way of introducing a cost of capital that varies with its riskiness, while not requiring an additional state variable. In other words, CARA utility here is used as a proxy for (i) and (ii) without the technical difficulties that either of these choices entails.

2 Model Solution

In this section, we solve the model and derive the optimal payout agreement. First, we derive the state variables. This requires us to analyze the intermediary's problem and apply Martingale-representation methods to its certainty equivalence continuation value, which will function as a state-variable in the tradition of Spear and Srivastava (1987). We then set up the dynamic program to be maximized. Tractability comes from collapsing the state-variables of the model to the one-dimensional difference between the intermediary's promised continuation value and the firm's cash holdings, resulting in an ODE. Finally, we derive the solution to the program. The solution consists of the optimal policies, i.e., the optimal risk-sharing, deferred payouts, and refinancing policies, the value function, as well as determining the (endogenous) lower boundary condition, which establishes which firms face the risk future liquidation.

2.1 State Variables

We now derive the three state variables of the problem: firm's cash holdings, intermediary's savings, and intermediary's continuation value. The intermediary's continuation utility U_t, a martingale in the utility space, can be expressed as W_t in certainty equivalent monetary terms, with

$$dW_t = \left[\frac{\rho r}{2}(\beta_t \sigma)^2 - \pi \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right] dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t \tag{10}$$

where α and $\sigma\beta$ are the loadings of dW on $d\Pi$ and dZ.[7] Optimal consumption is given by $c_t^* = rW_t$. Define the intermediary's continuation value derived from the contract with the firm as

$$Y \equiv W - S, \tag{12}$$

which is restricted to $Y \geq 0$ due to limited commitment by the intermediary. Note that liquidation forces $Y = 0$ by assumption. We defer discussion of implementation of the outside shareholders' limited commitment (LC) to after the reduction of the state variables. The dynamics of Y are

$$dY = dW - dS = \left[rY + \frac{\rho r}{2}(\beta_t \sigma)^2 - \pi \frac{1-e^{-\rho r \alpha_t}}{\rho r}\right] dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t - dI_t, \tag{13}$$

where we substituted in for the optimal consumption $c_t^* = rW_t$. Integrating up and imposing the transversality condition $\lim_{t\to\infty} e^{-rt} Y_t = 0$, we are left with

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} \left\{ dI_s - \left[\frac{\rho r}{2}(\beta_s \sigma)^2 + \pi\left(\alpha_s - \frac{1-e^{-\rho r \alpha_s}}{\rho r}\right)\right] ds \right\}\right]. \tag{14}$$

In words, the intermediary's continuation value from staying with the firm is simply the expectation of the cumulative discounted future transfers dI adjusted for dis-utility of risk-exposure, which is the term $[\cdot]\, ds$. We refer to Equation (13) as the *promise-keeping constraint.* It means that current transfers dI_t must be accompanied by a commensurate change in future promised transfers dY_t (i.e., $dY_t/dI_t = -1$) to deliver the promised value at time t (i.e., Y_t) to the intermediary.

Remark 1. *α (β) is the payoff, in terms of continuation value W, to the intermediary in response*

[7] The dynamics of the intermediary's continuation utility process, via martingale representation, are related to the contract's risk loadings β and α (appropriately scaled for convenience) as follows:

$$dU_t = rU_t dt - u(c_t)\, dt + \beta(-\rho r U_t)\, \sigma dZ_t + \left(e^{-\rho r \alpha_t} - 1\right) U_t \left(d\Pi_t - \pi dt\right). \tag{11}$$

Optimal consumption implies that $u'(c_t^*) = -\rho r U_t \iff u(c_t^*) = rU_t$, and thus U_t is a martingale. Applying Ito's Lemma to the certainty equivalent monetary wealth $W_t \equiv \frac{-\ln(-\rho r U_t)}{\rho r}$, we end up with (10).

to a refinancing opportunity $d\Pi = 1$ (Brownian cash-flow shock σdZ). If $\alpha > 0$ ($\beta > 0$), then the intermediary's continuation value W increase in response to a refinancing opportunity $d\Pi = 1$ (positive cash-flow shock $\sigma dZ > 0$). Whether Y increases depends on the difference $[\alpha_t - \alpha_I(t)]$ ($[\beta_t\sigma - \sigma_I(t)]$): if the difference is identically zero, then Y stays constant upon a refinancing shock (positive cash-flow shock), as the promised increase in Y is immediately cashed out via I. From (14), we see that any exposure $\alpha > 0$ ($\beta > 0$) requires a risk-premium of $\left(\alpha - \frac{1-e^{-\rho r\alpha}}{\rho r}\right) > 0$ ($\frac{\rho r}{2}(\beta\sigma)^2 > 0$) per dt to the intermediary.[8] Note that a contract without any intermediary financing is characterized by $\alpha_t = \beta_t = 0$ and consequently has $I_t = Y_t = 0$ for all t.

2.2 Dynamic Program

In the following section, we derive an expression for the value of outside equity that depends on the three endogenous state variables of the problem: the intermediary's certainty equivalent W_t, the intermediary's savings S_t, and the firm's cash holdings M_t. The intermediary's transfer process dI_t is then defined as the residual process that is consistent with the optimal (W_t, S_t, M_t) processes.

Reduction of the State Space. Because of the absence of wealth effects due to the intermediary's CARA utility, and the fact that the intermediary is not financially constrained, only Y_t instead of (W_t, S_t) matters. Let us define the variable C, something we term *excess liquidity*, by

$$C \equiv M - Y. \tag{15}$$

The dynamics of excess liquidity C are then given by

$$\begin{aligned} dC_t = dM_t - dY_t = &\left[\mu + (r-\lambda) C_t - \lambda Y - \frac{\rho r}{2}(\beta_t\sigma)^2 + \pi\frac{1-e^{-\rho r\alpha_t}}{\rho r}\right] dt \\ &+ \sigma(1-\beta_t) dZ_t + (C_t^* - C_t) d\Pi_t - dDiv_t, \end{aligned} \tag{16}$$

where we used $C_t^* \equiv M_t^* - Y_t^*$ with $M_t^* \equiv \Delta M_t + M_t$ and $Y_t^* \equiv Y_t + \alpha_t$.

Y as a choice variable. Note that C is *invariant* to transfers to/from the intermediary under promise keeping, and thus Y can be seen as a choice variable: asking the intermediary to provide a certain amount of cash to the firm requires an equivalent increase in its continuation value. As M and Y increase by the same amount, C remains unchanged.

[8] Let $h(\alpha) \equiv \alpha - \frac{1-e^{-\rho r\alpha}}{\rho r}$. Then $h(0) = 0$ and $h'(\alpha) = 1 - e^{-\rho r\alpha} > 0$ for $\alpha > 0$, so that $h(\alpha) > 0, \forall\alpha > 0$.

Importantly, even though Y is a choice variable, it is constrained. First, consider positive C. Then clearly the lowest possible Y is $Y = 0$ by the intermediary's limited commitment constraint. Next, consider negative C.[9] We require $Y \geq 0$ by the intermediary's limited commitment, and $M \geq 0$ by the technological constraint on cash. Then $Y = -C$ is the lowest possible Y — the one that applies for $M = 0$ — if the current state $C < 0$. Combining, we are left with the constraint

$$Y \geq \max\{0, -C\}. \tag{17}$$

Maximizing firm value. To translate the security design problem (7) into a standard dynamic programming problem, first note that maximizing the founder's payoff V_0 is equivalent to maximizing the initial net value of the firm, which is given by the sum of outside equity P_0 plus the intermediary's contribution $(-S_0)$ less the firm's initial cash endowment, M_0. Further, the intermediary's contribution occurs in exchange for equivalent future promises, i.e., $Y_0 = -S_0$, so that by (9) we have

$$\max_{\mathcal{C} \text{ is } IC} V_0 = \max_{\mathcal{C} \text{ is } IC} P_0 + Y_0 - M_0 = \max_{\mathcal{C} \text{ is } IC} P(C_0) - C_0. \tag{18}$$

where we used $C_0 = M_0 - Y_0$ and wrote the value of P_0 as a function of the state-variable C_0, $P(C_0)$. In words, the founder's problem is *equivalent* to maximizing the initial outside equity value less the initial net cash position. To solve this problem, we first dynamically maximize the outside equity value $P(C_0)$ for a given level of initial net liquidity C_0, and then determine the optimal level of initial net liquidity. We will show below that $P(C) \geq C$, so that the founder always extracts a non-negative payoff.

For the equilibrium value function $P(C)$, note that the *net value* (or surplus) $V(C)$ of the company, which is the sum of the value of the outsider shareholders and the value of the intermediary's claims *net* of cash is given by

$$V(C) \equiv P(C) + Y(C) - M(C) = P(C) - C. \tag{19}$$

Shareholders' limited commitment. Next, let us discuss possible restrictions imposed by the limited commitment (LC) of outside shareholders. Consider the change in value to the outside shareholders in case of a refinancing opportunity which moves the firm from some C to C^* with a prescribed α increase in continuation value to the intermediary. The change in value to the outside

[9] This corresponds to a situation in which the intermediary is willing to keep the firm out of liquidation. Then, it may be possible that $C = M - Y < 0$ as a perpetual firm may generate high Y even though current M is low.

investors is

$$P\left(C^{*}\right)-\Delta M-P\left(C\right)=P\left(C^{*}\right)-\left(C^{*}-C+\alpha\right)-P\left(C\right). \tag{20}$$

Note that $Y^{*}=Y+\alpha$ and $M^{*}=M+\Delta M$, so using the definition of C they need to raise an amount $\Delta M=C^{*}-C+\alpha$ in cash to transition from C to C^{*} in case of refinancing while delivering a payout α to the intermediary. Then LC restricts α as shareholders are assumed to always be able to vote down any equity issuance during refinancing, i.e., they may be unable to commit to specific refinancing policies. Thus, α is subject to the constraint

$$\alpha\leq P\left(C^{*}\right)-\left(C^{*}-C\right)-P\left(C\right). \tag{21}$$

Effectively, LC imposes that shareholders can never be made worse off by refinancing. This naturally implies $P\left(C\right)\geq 0$. Let $\mathcal{S}\left(C^{*},C\right)$ denote the set of all admissible α's.[10]

Outside Equity's HJB. Let us write the Markovian value function of the outsider shareholders, (8), as $P\left(C\right)$. We conjecture that the firm pays out dividends at an upper boundary $C=\overline{C}$ and that it either liquidates or receives sufficient financing to stay alive at some lower boundary $\underline{C}$. We can now write the outside equity valuation HJB w.r.t. C on $\left(\underline{C},\overline{C}\right)$ as

$$\begin{aligned} rP\left(C\right)=&\max_{\beta,Y\geq\max\{0,-C\}}\left\{P'\left(C\right)\left[\mu+\left(r-\lambda\right)C-\lambda Y-\frac{\rho r}{2}\left(\beta\sigma\right)^{2}\right]+P''\left(C\right)\frac{\sigma^{2}}{2}\left(1-\beta\right)^{2}\right\}\\ &+\pi\max_{C^{*},\alpha\in\mathcal{S}\left(C^{*},C\right)}\left\{P'\left(C\right)\frac{1-e^{-\rho r\alpha}}{\rho r}+\left[P\left(C^{*}\right)-P\left(C\right)-\left(C^{*}-C+\alpha\right)\right]\right\}. \end{aligned} \tag{22}$$

The appropriate boundary conditions at $C=\overline{C}$ and $\underline{C}$ are discussed after the derivation of the optimal policies on $\left(\underline{C},\overline{C}\right)$ as they interact.

2.3 Solution

Optimal policies. From the HJB, observe that the policy functions α,β,C^{*},Y do not directly interact with each other except that C^{*} affects the admissible set of α. However, we observe that that the constraint on α is relaxed by picking a higher C^{*}. As we will see below, the optimal C^{*} is the highest of all feasible C^{*}, and thus the constraint on α is essentially *static*.

[10] In the absence of LC on the refinancing decision, we still have the limited commitment restriction $P\left(C\right)\geq 0$. This in turn requires the restriction $\alpha\leq P\left(C^{*}\right)-\left(C^{*}-C\right)$, which will only be binding for low C. With full commitment by the shareholders, they choose $\alpha_{U}(C)$ throughout which can lead to $P(C)<0$ for low C This latter situation is hard to map into the real world as $\alpha_{U}(C)$ can surpass the maximum that can be raised via equity issuance.

Y: As $P'(C) > 0$ and $\lambda > 0$, the contract picks the lowest Y possible subject to constraint (17), so that

$$Y(C) = \max\{-C, 0\} \quad \text{and} \quad M(C) = \max\{C, 0\}. \tag{23}$$

In words, it does not pay to defer payouts for $C > 0$, and it does not pay to hold cash inside the firm for $C < 0$. The left panel of Figure 1 illustrates. Effectively, when $Y(C) > 0$ the firm is financed via promises, while when $M(C) > 0$ the firm is financed via cash.

C^*: The FOC w.r.t. to the refinancing target C^* yields

$$P'(C^*(C)) = 1, \tag{24}$$

so that refinancing occurs up until a point at which the internal value of cash is equalized with the value of paying it out. Note that $P'(C) \geq 1$ by the simple fact that dividend payouts are always available.

α: The FOC w.r.t. the refinancing payout to the intermediary α yields different policies depending on whether LC is binding or not. Let α_U be the optimal policy when the constrained is not binding, and let $\alpha_{LC}(C)$ be the optimal policy when the constraint is binding. Then we have

$$\alpha_U(C) \equiv \frac{\log P'(C)}{\rho r} \quad \text{and} \quad \alpha_{LC}(C) \equiv P(C^*) - C^* - [P(C) - C], \tag{25}$$

and the optimal α is given by

$$\alpha(C) = \min\{\alpha_U(C), \alpha_{LC}(C)\}. \tag{26}$$

We will show below that LC is always binding in equilibrium, i.e., $\alpha(C) = \alpha_{LC}(C) < \alpha_U(C)$. The right panel of Figure 1 illustrates the difference between $\alpha_U(C)$ (dotted red line) and the actual $\alpha(C) = \alpha_{LC}(C)$ (solid black line) for parameters given in Table 1.

β: The FOC w.r.t. instantaneous risk-sharing β yields

$$\beta(C) = \frac{P''(C)}{P''(C) - \rho r P'(C)} \in [0, 1]. \tag{27}$$

Note $A(C) \equiv -\frac{P''(C)}{P'(C)}$ is the effective risk-aversion of the outside investors, so risk-sharing with the intermediary is higher the higher the effective risk-aversion of the outside investors

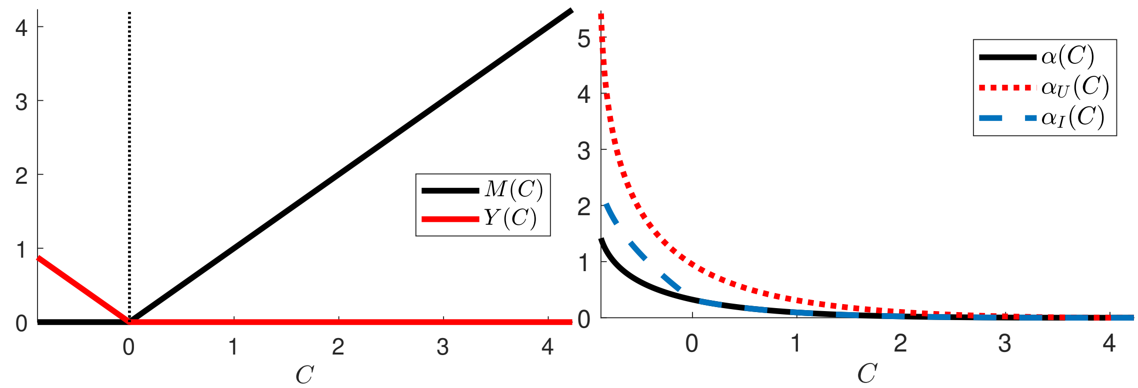

Figure 1: **Left Panel**: Numerical illustration of the optimal cash-holdings $M(C) = \max\{C, 0\}$ and deferred payouts $Y(C) = \max\{-C, 0\}$ as a function of the state variable. **Right Panel**: Numerical illustration of optimal α (solid black; here *limited liability*), the unconstrained $\alpha_U(C)$ (dotted red) and the total actual payout upon refinancing $\alpha_I(C) = \alpha(C) + Y(C)$ (dashed blue).

as $\beta(C) = \frac{1}{1+r\frac{\rho}{A(C)}}$. Then $A(C) \to 0$ implies $\beta(C) \to 0$, and $A(C) \to \infty$ implies $\beta(C) \to 1$.

Upper boundary $\overline{C}$. The upper (or dividend payout) boundary satisfies the standard *smooth pasting* and *super contact* conditions

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \tag{28}$$

Note $P'(C) = 1$ for $C \geq \overline{C}$, as well as $P''(C) < 0$ from the ODE for $C < \overline{C}$. Then, the concavity of the outside equity value reflects that the outside investors become effectively more risk averse due to the possibility of financial distress when the firm's liquidity reserves dwindle.

The boundary conditions imply $\alpha\left(\overline{C}\right) = \beta\left(\overline{C}\right) = 0$, i.e., there is no risk-sharing at the dividend payout boundary, and $C^*(C) \geq \overline{C}$, so wlog we set $C^*(C) = \overline{C}$ and refinancing is always to the payout boundary.[11]

Note that at the outset of the problem, we would not have been able to rule out the following scenario: due to the LC restriction on α, the firm might optimally refinance to below the payout boundary, $C^* < \overline{C}$. But we now see from the optimal policies above that there is no trade-off between C^* and α in that a higher C^* allows for more states of unrestricted α. Thus, the firm always refinances to the boundary $C^* = \overline{C}$ and restricts α as required.[12]

[11] Any $C^* > \overline{C}$ fulfills the FOC but leads to an immediate payout $C^* - \overline{C} > 0$ – essentially the firm would raise cash just to immediately pay it back to its shareholders. Setting $C^* = \overline{C}$ minimizes these roundtrip transactions.

[12] Due to $P'(C) \geq 1$ the firm is always in a situation to raise enough cash to achieve any $C^* \in \left(C, \overline{C}\right]$, possibly by cutting down on α.

Let Y^A denote the *autarky value* to the intermediary of running the firm forever, i.e.,

$$Y^A \equiv \frac{\mu}{r} - \frac{\rho}{2}\sigma^2. \tag{29}$$

Then $\max\left\{Y^A, 0\right\}$ is a lower bound on total firm value at the payout boundary – if it were not, the firm could simply sell itself to the intermediary for $Y^A > 0$ or liquidate if $Y^A < 0$, paying out all cash C and any sales proceeds to the shareholders. Plugging the optimal policies (23)-(27) into the HJB (22), the total value of the firm (19) at the payout boundary $\overline{C}$ is given by

$$V\left(\overline{C}\right) = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} \geq \max\left\{Y^A, 0\right\} \iff \min\left\{\frac{\mu}{\lambda}, \frac{\rho r}{2\lambda}\sigma^2\right\} \geq \overline{C}. \tag{30}$$

This holds for *any* feasible contract with an optimal payout boundary, even for the no-intermediary-financing contract $\alpha = \beta = 0$.

Next, the firm's NPV in a frictionless market is given by $\frac{\mu}{r}$, and thus we must have $\overline{C} \geq 0$ for any $\lambda > 0, \rho > 0$, as the presence of frictions cannot increase the value of the firm $V(C)$ beyond its frictionless NPV value. Thus, at the payout boundary, the total value of the firm is simply its frictionless NPV less the discounted carry cost of holding cash $\overline{C}$ for an infinite duration, and the efficiency of the contract is reflected in the level of $\overline{C}$ (with lower being better).

Plugging in the optimal policies and the boundary value, we are left with the simplified HJB

$$\begin{aligned}(r+\pi)\,P(C) =& P'(C)\left[\mu + \left(r - \lambda \mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\,\sigma^2 + \pi\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right] \\ &+ \pi\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + C - \alpha(C)\right].\end{aligned} \tag{31}$$

Endogenous lower boundary $\underline{C}$. Next, does the firm always liquidate when cash runs out, i.e., at $M = 0$? This is equivalent to asking if the intermediary optimally provides financing at the lower boundary $\underline{C}$ to stave of firm liquidation or not. After plugging in the optimal policies, we can express the drift of C as

$$\mu_C(C) = \mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\sigma^2\beta(C)^2 + \pi\frac{1-e^{-\rho r\alpha(C)}}{\rho r}. \tag{32}$$

For $\underline{C}$ to be a lower bound C, it must be that either (i) the firm liquidates at $\underline{C}$ or that (ii) $\underline{C}$ is either a reflection, inaccessible, or an absorbing state (absent refinancing) in which the intermediary is happy to run the firm by itself.[13] Let us consider both cases in turn:

(i). Recall we assumed the liquidation value (beyond cash) to be zero, so liquidation imposes $Y = 0$, while we derived that optimally $Y(C) = \max\{0, -C\}$. Thus, if liquidation occurs, it must occur on $\left[0, \overline{C}\right]$ as otherwise promise keeping is violated. Next, liquidation at any $C = \underline{C}^{\text{liquidation}} > 0$ with dividend C to the shareholders is sub-optimal, as $P(C) > C$ for any $C > 0$ by $P'(C) > 1$. Thus, conditional on liquidating, it is optimal to liquidate at the lowest value C not violating promise keeping,

$$\underline{C}^{\text{liquidation}} = 0. \tag{33}$$

(ii). For $\underline{C}$ to be a lower bound of C absent liquidation, something we will term *survival*, it must be that C's volatility vanishes, while its drift stays non-negative as $C \to \underline{C}$.[14] Intuitively, we want to delay the inefficient full takeover of risk by the intermediary as long as possible. We prove the following Lemma in the Appendix:

Lemma 1. *Let $w(\cdot)$ be the product logarithm or Lambert's w-function. Conditional on survival, for a given payout boundary $\overline{C}$, the optimal $\underline{C}^{survival}$ is given by*

$$\underline{C}^{survival}\left(\overline{C}\right) = \frac{w\left(\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right\}\right) - \frac{\pi}{r}}{\rho r} - Y^A. \tag{35}$$

Further we have $\lim_{\pi\to\infty} \underline{C}^{survival}\left(\overline{C}\right) = -\frac{\mu}{r}$ *and* $\frac{\partial \underline{C}^{survival}\left(\overline{C}\right)}{\partial \pi} < 0.$

For any fixed $\overline{C}$, we now have to determine the optimal policy amongst (i) liquidation and (ii) survival: First, consider $\underline{C}^{\text{survival}} < 0$. Suppose the firm were to liquidate at $C = 0$. This is sub-optimal as it would ensure $P(0) = 0$ and $Y(0) = 0$, whereas survival at $\underline{C}^{\text{survival}}$ yields the same $Y(0) = 0$ but a higher $P(0) > 0$. Next, consider $\underline{C}^{\text{survival}} > 0$. At this absorbing state (absent refinancing), the outside investors have a value $P(\underline{C}^{\text{survival}}) = 0$ while the intermediary has a value of $Y(\underline{C}^{\text{survival}}) = 0$ from running the firm. To make the intermediary just willing to run the

[13] A similar approach to determining the lower boundary is used in Bolton et al. (2019). A reflection is not possible due to promise keeping and the inability to access outside cash not from the intermediary: C is invariant to Y, so there is no impulse control possible.

[14] That is, we require $\sigma_C(\underline{C}) = 0$, which requires $\beta(\underline{C}) = 1$, and $\mu_C(\underline{C}) \geq 0$. This in turn implies $\lim_{C\to\underline{C}} P''(C) = -\infty$:

$$1 = \lim_{C\to\underline{C}} \beta(C) = \lim_{C\to\underline{C}} \frac{P''(C)}{P''(C) - \rho r P'(C)} = \lim_{C\to\underline{C}} \frac{1}{1 - \rho r \frac{P'(C)}{P''(C)}} \iff \lim_{C\to\underline{C}} \frac{P'(C)}{P''(C)} = 0. \tag{34}$$

As $P'(C) \geq 1$, we require $\lim_{C\to\underline{C}} P''(C) = -\infty$, and $P''(C)$ diverges faster than $P'(C)$ in case $P'(C)$ diverges.

firm in autarky until a refinancing event, i.e., $Y(\underline{C}^{\text{survival}}) = 0$, a cash amount $M = \underline{C}^{\text{survival}} > 0$ is required to generate interest payments to add to the cash-flows from assets. However, outside investors could be made strictly better off by paying out $\underline{C}^{\text{survival}}$ as dividend and liquidating the firm immediately afterwards, while the intermediary would remain at $Y(\underline{C}^{\text{survival}}) = 0$. But we can do even better, as liquidation at any $C > 0$ is sub-optimal compared to liquidation at $C = 0$. Consequently, the lower boundary is given by

$$\underline{C}\left(\overline{C}\right) = \min\left\{\underline{C}^{\text{survival}}\left(\overline{C}\right), \underline{C}^{\text{liquidation}}\right\} = \min\left\{\underline{C}^{\text{survival}}\left(\overline{C}\right), 0\right\}, \tag{36}$$

with associated boundary condition

$$P\left(\underline{C}\right) = 0. \tag{37}$$

How can $P\left(\underline{C}\right) = 0$ hold for a firm that never liquidates, as there is only positive dividends in the future as limited liability holds? Upon refinancing, the existing outside shareholders are completely diluted, while the intermediary receives the payment $\alpha_I\left(\underline{C}\right)$. Any additional restriction – say the limited commitment upon refinancing – shrinks the maximum credible promise vis-a-vis a case of commitment by shrinking $\overline{C}$.

To build intuition for a firm's risk of liquidation, consider the case of permanently shut equity markets, i.e., $\pi = 0$. For $Y^A < 0$ the firm will be liquidated when it runs out of cash, i.e., $M = 0$, while for $Y^A > 0$ the firm is optimally kept alive when it runs out of cash by the intermediary on $C < 0$ with an absorbing state at $\underline{C} = -Y^A$. That is, at $C = \underline{C}$, the intermediary has fully taken over the firm, and having no avenue to sell it, derives the autarky value Y^A. Thus, this is the most that can be credibly promised to the intermediary in the absence of refinancing. As π increases and refinancing opportunities arise, some firms are optimally kept alive even for negative autarky values Y^A. This comes about as refinancing provides an exit for the intermediary to reallocate its stake to investors with a higher, risk-neutral, valuation.

Optimal transfer process dI. Lastly, let us determine dI, the equilibrium transfer process. For $C \geq 0$, we optimally have $Y = 0$ which implies $dY = 0$. Thus, by the expression for dY in (13),

$$dI = \left[\frac{\rho r}{2}\left(\beta_t \sigma\right)^2 - \pi \frac{1 - e^{-\rho r \alpha_t}}{\rho r}\right] dt + \beta_t \sigma dZ_t + \alpha_t d\Pi_t \text{ for } C \geq 0. \tag{38}$$

Note that $\sigma_I = \sigma\beta = \sigma_W$, and the volatility of dW and dI coincide for $C > 0$. This is intuitive, as there is no change in deferred payouts $Y = 0$, and thus all changes in W lead to equivalent cash-flows, i.e., they must coincide with changes in I.

For $C < 0$, we optimally have zero cash, i.e., $M = C + Y = 0$. Absent refinancing, $dM = dC + dY = 0$, while upon a refinancing opportunity $C^*(C) + Y^*(C) = \overline{C} > 0$,[15] which implies

$$dI = \mu dt + \sigma dZ_t + (\alpha_t + Y_t)\, d\Pi_t \text{ for } C < 0. \tag{40}$$

In words, on $C < 0$ the intermediary completely absorbs any cash-flow shocks $dX_t = \mu dt + \sigma dZ_t$, while gaining $\alpha_t + Y_t$ upon refinancing. Note that on $C \in (\underline{C}, 0)$, we have $\beta(C) < 1$ and therefore $\sigma_W = \beta\sigma < \sigma = \sigma_I$, i.e., the volatility of dW and dI diverge as part of the continuation value is delivered via deferred payouts, i.e., changes in Y. The refinancing payout $\alpha_I(C) = \alpha(C) + Y(C) \geq \alpha(C)$ can be larger than the change in W as Y is optimally paid out during refinancing. The dashed blue line in the right panel of Figure 1 illustrates α_I for parameters given in Table 1.

Solution. We are now in a position to characterize the optimal financing arrangement:

Proposition 1. *The optimal contract is described by the optimal policies (23), (24), (27), (26), the optimal transfer process characterized by (38) and (40), and the outside share-holder's value function solves the ODE (31) with associated boundary conditions*

$$P'(\overline{C}) - 1 = P''(\overline{C}) = P(\underline{C}) = 0, \tag{41}$$

where $\underline{C}$ is given by (36). The outside share-holder's value function is concave, i.e., $P''(C) \leq 0$, and its slope is at least unity, i.e, $P'(C) \geq 1$.

Next, at the payout boundary $\alpha\left(\overline{C}\right) = \beta\left(\overline{C}\right) = 0$. If $\underline{C} = 0$, the firm liquidates at $C = \underline{C}$ with $\beta(\underline{C}) < 1$, while a firm with $\underline{C} < 0$ will never liquidate and $\beta(\underline{C}) = 1$. Further, both $\overline{C}$ and $\underline{C}$ are decreasing in π, with limits 0 and $-\frac{\mu}{r}$, respectively.

Lastly, LC implies that that α is always constrained, i.e.,

$$\alpha(C) = \alpha_{LC}(C) = P(C^*) - C^* - [P(C) - C]. \tag{42}$$

[15] The process $dI = \mu_I(t)\, dt + \sigma_I(t)\, dZ_t + \alpha_I(t)\, d\Pi$ is defined as the residual that makes the following hold:

$$dM = C^*(C)\, d\Pi = dC + dY = [\mu - \mu_I(r)]\, dt + [\sigma - \sigma_I(t)]\, dZ_t + [C^*_t - C_t + \alpha_t - \alpha_I(t)]\, d\Pi. \tag{39}$$

and both $\alpha(C)$ and $\beta(C)$ are monotonically decreasing in C.

We present numerical examples based on the common parameters given in Table 1. The value function $P(C)$ for $\pi = 0$ (left panel) and $\pi = 0.25$ (right panel) are depicted as the solid black lines in the top row of Figure 2, the payout boundary $\overline{C}$ as the vertical red lines, and lower boundary $\underline{C}$ as the vertical blue lines. We see that in the left panel for $\pi = 0$, we have $\underline{C} = 0$ as $Y^A < 0$, and the firm is liquidates once it runs out of cash. Contrast this with the right panel for $\pi = 0.25$, so the expected time until market access is $1/\pi = 4$ years. As $\underline{C} < 0$ indicates, the firm never liquidates as the intermediary is willing to provide financing against promises Y.

Parameter	Value	Interpretation
r	0.06	Common discount & interest rate
λ	0.01	Internal carry cost of cash
μ	0.18	Drift of cash-flow process
σ	1.5	Volatility of cash-flow process
ρ	6	Coefficient of absolute risk aversion
$Y^A = \frac{\mu}{r} - \rho\frac{\sigma^2}{2}$	-3.75	Autarky value of firm to intermediary

Table 1: Baseline Parameter Values for all Figures.

Revisiting the initial cash-endowment. By the concavity of $P(C)$ and the fact that $P'(C) = 1$ for $C \geq \overline{C}$, the firm's optimal *initial* level of net liquidity C_0 chosen by the founder coincides with the dividend payout boundary $\overline{C}$, i.e.,

$$\overline{C} = \arg\max_{C_0 \geq 0} P(C_0) - C_0. \tag{43}$$

Renegotiation proofness. Note that all contracts derived above are renegotiation proof – given any future renegotiation time $\hat{\tau}$ with some $C_{\hat{\tau}}$, the same contract as derived above is picked.[16] This can be seen through two observations: First, external cash cannot be raised *outside* a refinancing opportunity, and thus C can only be moved down: either any reshuffling between M and Y leads to the same C, or the firm is giving out free promises Y, lowering C which costs $P'(C) > 1$ for only a return of 1 to the intermediary. Further, the contract above already picked the optimal Y to minimize the cost of holding cash, i.e., (23). Thus, there exists no reshuffling of the delivery of Y

[16]We follow Strulovici (2020) in defining renegotiation proofness in the presence of an exogenous state, here C.

that would deliver such promises in a cheaper way, i.e., α, β are already optimal. Second, consider renegotiation *during* a refinancing opportunity. With external cash available, we should renegotiate as long as surplus can be increased, i.e., $(P(C^*) + Y(C^*) - M(C^*))' = (P(C^*) - C^*)' > 0$. But as (24) implies we are always refinancing to the payout boundary, which maximizes surplus.

As an important implication, today's contract with the current intermediary is thus not affected by a possible switch to and renegotiation with a new intermediary (given appropriate side-payments of value Y between the intermediaries) at a future time.

3 Contract dynamics and implementation

In this section we discuss the contract dynamics, and then present an implementation. First, we discuss the optimal financing policies in greater detail, and show comparative statics. Some of this discussion will preview the preferred implementation. Next, we introduce our preferred implementation of the optimal contract with common financial instruments, here a credit line from the intermediary, cash holdings by the firm, common equity to outside investors, and restricted equity to the intermediary. We argue that this implementation is natural by establishing tight links of the contract to the credit line, common equity, as well as restricted equity.

3.1 Contract dynamics

Here, we discuss the optimal financing policies, as they determine the contract dynamics. All numerical examples are based on the common parameter values given in Table 1.

Continuous financing β. As β is smoothly decreasing in C, when C is low, the contract has high instantaneous risk-sharing β between the firm and the intermediary, leading to lower volatility of C. In other words, the firm relies heavily on covering cash short-falls with intermediary funds, but at the same time repays the intermediary rapidly for cash inflows. Such financing is costly as the risk-premium term in dW (10) indicates, and thus lowers the drift of C. Consequently, β optimally trades off volatility and drift of C.

The bottom row of Figure 2 depicts the instantaneous financing policies $\beta(C)$ for $\pi = 0$ (left panel, solid black line) and $\pi = 0.25$ (right panel, dashed red line). We see that the left panel features liquidation as $\underline{C} = 0$ and $\beta(\underline{C}) < 1$, while the right panel features survival as $\underline{C} < 0$ and $\beta(\underline{C}) = 1$. Note the divergence of $\sigma\beta(C_t)$ (dashed red line) from σ_t^I (solid black line), both

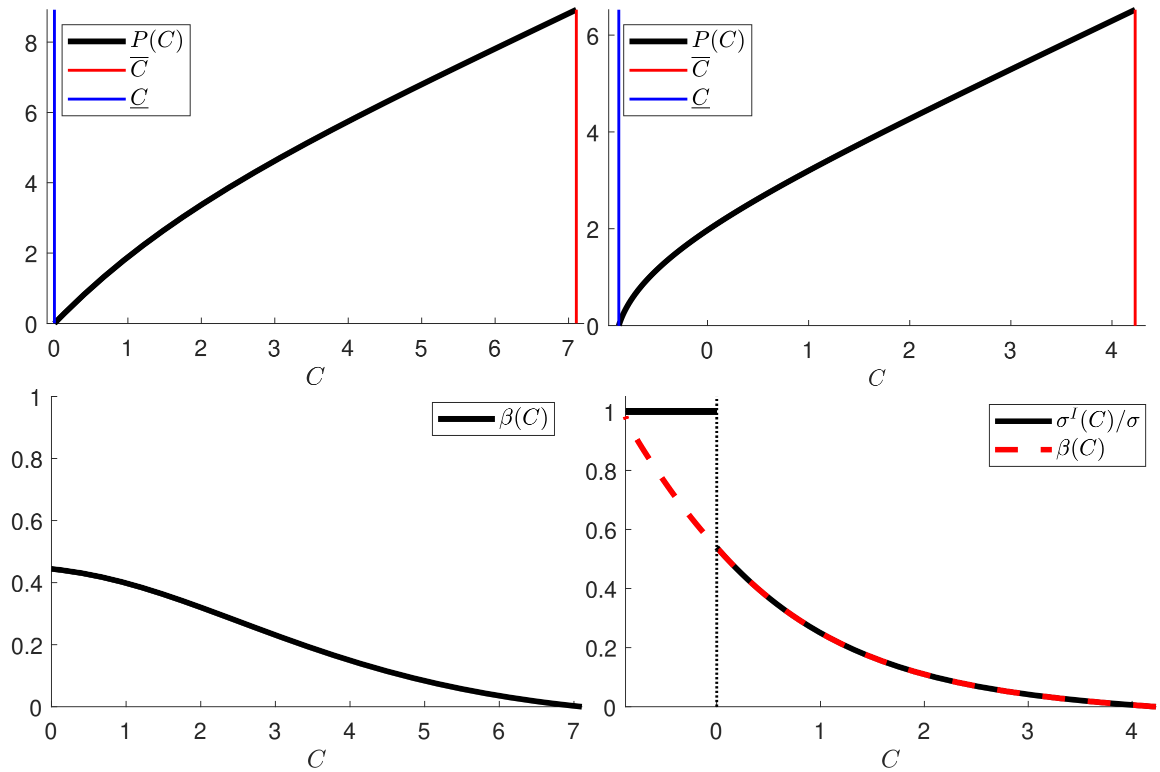

Figure 2: Numerical illustration of the value function and optimal cash flow risk borne by the intermediary $\beta(C)$. The parameters are such that $0 = Y^A$, and consequently the firm is liquidated at $C = 0$ for $\pi = 0$ (**left panels**) but not for $\pi > 0$ (**right panels**). The parameters are $r = 0.06$, $\lambda = 0.01$, $\mu = 0.18$, $\pi = 0.25$, $\sigma = 1.5$, $\rho = 6$.

normalized by σ, for $C < 0$ that was discussed in Section 2.3.

Refinancing payout α**.** As α is smoothly decreasing in C, when $\pi > 0$ and refinancing opportunities exist, the intermediary optimally transforms some flow payout today into a (promised) lumpy payout upon refinancing. This increases the drift of C in (16), but due to the intermediary's risk-aversion is not an "actuarially fair" exchange: it requires a larger payout increase α than it improves the drift of C in expectation. The optimal contract nevertheless uses this instrument as the outside shareholders are increasingly (effectively) risk-averse as C decreases. Consequently, α optimally trades off the jump component and drift of C.

The right panel of Figure 1 depicts $\alpha(C)$ (solid black line) when $\pi = 0.25$. The dotted red line depicts the unconstrained optimal $\alpha_U(C)$. As we proved, α is always constrained by LC and thus there is a gap between the two lines. We will return to the interpretation of the gap when we discuss our preferred implementation.

Deferred payouts Y and cash-holdings M. The optimal contract minimizes the deferral of payouts to the intermediary, i.e., it picks the lowest possible Y as shown in (23), as depicted in the left panel of Figure 1.

For $C > 0$ we have positive cash-holdings $M > 0$, and increasing Y requires an equivalent increase in M by promise keeping to keep C constant. But by our assumptions, holding cash inside the firm is inefficient at a rate $\lambda > 0$. Thus, it is optimal to have the minimum amount of cash consistent with $C > 0$, which yields $Y = 0$.

Next, we have $\underline{C} < 0$ if and only if the intermediary is willing to temporarily (for $\pi > 0$) or permanently (for $\pi = 0$) assume all cash-flow risk. Suppose this is the case. Then, for $C < 0$, the firm compensates the intermediary with deferred payouts only and holds no cash reserves, i.e., $Y > 0$ but $M = 0$. Consider shut down equity markets, i.e., $\pi = 0$, which implies $\underline{C} = -Y^A$. Thus, for a bad enough sequence of shocks, the deferred payouts approaches the absorbing state Y^A, which corresponds to permanent ownership of the firm by the intermediary. An intuitive interpretation of deferred payouts Y is therefore that of an equity stake. With $\pi > 0$, $Y(C)$ incorporates both the holding value of the cash-flow stream to the intermediary as well as the resale value to risk-neutral investors, i.e., the exit value.

3.2 Implementation

Note that the security issued to the outside investors is common equity, and thus only the implementation of the contract between the firm and the intermediary remains. First we introduce the auxiliary function $T(C)$ to record cumulative transfers between the firm and the intermediary since last refinancing. Absent LC, $T(C)$ can be interpreted as the balance on a credit line and the equity stake, as it is *exactly* equal to the amount that is paid back to the intermediary upon refinancing. This insight guides us to implementing the contract with a credit line and an equity stake with LC. Interestingly, LC induces an "early repayment incentive" of the credit line upon refinancing: the balance is only partially repaid while the remainder is forgiven by the intermediary to provide refinancing incentives to the shareholders. The intermediary is compensated for this partial non-repayment with a higher interest rate on the credit line. Lastly, we solve for the common equity issuance to implement the refinancing.

3.2.1 Auxiliary function: Cumulative transfers since refinancing

Recall the transfer process dI is given by (38) and (40). Let $T(C)$ be the Markovian accounting function that records cumulative transfers due to dZ *since last refinancing*. By Ito's Lemma

$$\begin{aligned} dT(C) =& \mu_T(C)\,dt + \sigma_T(C)\,dZ_t - \alpha_T(C)\,d\Pi_t \\ =& \left[T'(C)\,\mu_C(C) + \frac{1}{2}T''(C)\,\sigma_C^2(C)\right]dt + T'(C)\,\sigma_C(C)\,dZ_t + \left[T\left(\overline{C}\right) - T(C)\right]d\Pi_t \end{aligned} \tag{44}$$

As the process has to reset upon refinancing due to its Markovian nature, we impose $T\left(\overline{C}\right) = 0$. Note $T(C_t)$ is subtly different from $-I_t$, as I_t turns out to be non-Markovian.[17]

Volatility Loading. For $T(C)$ to record cumulative contributions due to dZ, its volatility loading must match the negative of the transfer process dI's loading. In other words, total contributions increase (decrease) 1-for-1 with transfers from (to) the intermediary caused by dZ. Matching volatilities, and using (38), we have

$$\sigma_T(C) = -\sigma_I(C) \iff T'(C) = -\frac{\beta(C)}{1-\beta(C)} - \mathbf{1}_{\{C<0\}}. \tag{45}$$

Plugging in $\beta(C)$ from (27), integrating, and imposing $T\left(\overline{C}\right) = 0$, we have

$$T(C) = \frac{\log P'(C)}{\rho r} + Y(C) = \alpha_U(C) + Y(C). \tag{46}$$

Drift loading. Due to the non-Markovian nature of dI, there is no guarantee that imposing $\sigma_T + \sigma_I = 0$ implies matching drifts, i.e., $\mu_T(C) + \mu_I(C,t) = 0$. Noting that $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$, after deriving $\beta'(C)$ and some algebra, we can show that under optimal β risk-sharing we have

$$\mu_T(C) + \mu_I(C,t) = \frac{\lambda}{\rho r}\mathbf{1}_{\{C>0\}} + rY(C) + \pi\frac{e^{\rho r[\alpha_U(C)-\alpha(C)]} - 1}{\rho r}. \tag{47}$$

The RHS is made up of three terms: The first term reflects a constant drift part that is linked to the carry-cost-of-cash, λ. The second term reflects the cost of delaying the payout of the equity stake, which is linked to the discount rate. The third term reflects the additional compensation required to the intermediary from the LC restriction on α. In any Markov implementation, interest rates and other fees have to absorb this difference.

[17] For example, past refinancing events are recorded in I_t, but not in $T(C_t)$.

Refinancing loading. As the firm always refinances to the payout boundary $\overline{C}$, and $T\left(\overline{C}\right) = 0$ by definition, the jump loading is simply given by

$$\alpha_T\left(C\right) = T\left(C\right). \tag{48}$$

3.2.2 Credit line

Absent the LC constraint, the optimal contract has $\alpha(C) = \alpha_U(C)$, which would imply $\alpha_I(C) = \alpha_U(C) + Y(C) = T(C)$. In other words $T(C)$ would *exactly* equal to the actual payment upon refinancing, leading us to a credit line and equity stake interpretation: here, $Y(C)$ is the current equity stake, so that leaves $D(C) = \alpha_U(C)$ as the current balance of the credit line.[18]

When the LC constraint is imposed, the payoff to the intermediary upon refinancing is given by $\alpha_I\left(C\right) = \alpha(C) + Y(C)$. Note that as $\alpha\left(C\right) < \alpha_U\left(C\right)$ for $C < \overline{C}$, we have $\alpha_I\left(C\right) < T(C)$ for $C < \overline{C}$. In other words, the contract specifies that the intermediary optimally demands payment of *less* than its total cumulative transfers since last refinancing. This is due to incentives: demanding the full repayment $\alpha_U(C) + Y(C)$ violates LC, and thus will be vetoed by the shareholders. As discussed above, we will take the credit-line balance to be $D(C) = \alpha_U(C)$, and thus require "early repayment incentives" of amount $\alpha_U(C) - \alpha(C) > 0$, while $Y(C)$ is separately generated via restricted equity.

Any implementation via a general credit line balance of $D\left(C\right)$ together with a portfolio of other *non-interest* bearing instruments — in our interpretation restricted equity $Y(C)$ — by construction must have volatility matching $T(C)$. However, the drifts of I and T do not cancel due to I not being Markovian. To balance the drifts, i.e., match the dt dynamics, we introduce the Markovian interest rate $r_D\left(C\right)$ on the balance $D(C)$, as well as a constant maintenance fee f so that

$$\mu_T\left(C\right) + \mu_I\left(C, t\right) = r_D\left(C\right) D\left(C\right) + f. \tag{49}$$

We set f to absorb any constant payouts at $C = \overline{C}$, and let the interest rate $r_D\left(C\right)$ capture the remaining variable difference.[19] At the payout boundary we have $\alpha_U(\overline{C}) = \alpha(\overline{C}) = Y(\overline{C}) = 0$, so

$$f = \frac{\lambda}{\rho r} \quad \text{and} \quad r_D\left(C\right) = \frac{rY\left(C\right) - \frac{\lambda}{\rho r}\mathbf{1}_{\{C<0\}}}{D\left(C\right)} + \frac{\pi\frac{e^{\rho r\left[\alpha_U(C) - \alpha(C)\right]}-1}{\rho r}}{D\left(C\right)} \tag{50}$$

[18]Importantly, maintaining that the credit-line balance records net-transfers coming from credit line usage, $Y(C)$ has to be generated from payments separate from the credit line.

[19]Note that $\mu_T(C)$ summarizes all non-interest movements in $T(C)$, which by assumption our portfolio of instruments, here credit line and restricted equity, replicates.

Note that the *state-dependent* interest rate $r_D(C)$ has a jump at $C = 0$ as the equity stake $Y(C)$ enters the picture and the inefficiency of internal cash-holding λ disappears.[20] We see that the second term of $r_D(C)$ reflects the LC restriction on α, as it records the (risk-adjusted) required compensation for the early repayment incentive of $\alpha_U(C) - \alpha(C)$ upon refinancing. Further, we can show that $\lim_{C\to\overline{C}} r_D(C) = \pi$, so the impact of the credit-line's early repayment incentives – occuring at rate π – first-order dominates the discount rate r at the payout boundary.

3.2.3 Restricted equity

Recall that we argued in Section 3.1 that $Y(C)$ can naturally be understood as an equity stake. Let us construct the terms of this stake. For an equity stake to be used, we need $\underline{C} < 0$. Suppose that the firm allows the intermediary to trade equity internally at the given price schedule $P_I(C)$, but only allows equity shares to trade on the open market at refinancing opportunities.In essence, we will interpret the intermediary's equity shares as restricted equity that vests upon refinancing. As $D(C) = \alpha_U(C)$, the intermediary must $g_E(C)$ restricted equity shares to achieve the payout of $Y(C)$ upon financial market access through sales at post-issuance share-price $P_E(C)$:

$$Y(C) = g_E(C)\, P_E(C) \iff g_E(C) = \frac{Y(C)}{P(C)} \tag{51}$$

where we used $P_E(C) = P(C)$ due to LC.[21] Further, the portfolio of credit line and restricted equity, absent fees and interest, has to match the cumulative transfers, $T(C)$. As $T(C) = \alpha_U(C) + Y(C)$, and the credit line covers $D(C) = \alpha_U(C)$, the value $Y(C)$ has to be generated by the intermediaries' "trading" gains and losses of restricted equity, that is

$$Y(C) = \int_C^0 \left(-g'_E(x)\right) P_I(x)\, dx \tag{52}$$

Setting both expressions for $Y(C)$ equal and differentiating pins down $P_I(C)$:

$$\left(g_E(C)\, P(C)\right)' = -1 = g'_E(C)\, P_I(C) \iff P_I(C) = -\frac{1}{g'_E(C)}. \tag{53}$$

[20] By L'Hopital's rule, the interest rate on the credit line at the payout boundary is positive, and given by $r_D(\overline{C}) = \pi$ as $\alpha''(\overline{C}) = 0$ but $\alpha''_U(\overline{C}) \neq 0$. This is intuitive – the intermediary is approximately risk-neutral towards the very small loss of $\alpha_U(C) - \alpha(C)$, and thus is only compensated for the arrival rate of this loss, π.

[21] See next subsection for details.

Further, writing out $(g_E(C)\,P_E(C))'$ and dividing through by $g_E'(C)$, we see that

$$P(C) - \left(\frac{g_E(C)}{-g_E'(C)}\right) P'(C) = P_I(C)\,. \tag{54}$$

Because $g_E'(C) < 0$ and $P'(C) > 0$, the internal price is always below the external price, resulting in strict incentives for the intermediary to sell its shares on the open market upon refinancing.

3.2.4 Refinancing via common equity issuance

Next, we investigate the details of the refinancing. First, we normalize the current number of outstanding shares to unity, and let g be the number of new shares issued upon refinancing.[22] Then, the post-issuance equity price is given by $P_E(C) = \frac{P(\overline{C})}{1+g}$. As the proceeds from g must cover the total cash needed to replenish the firm's cash-holdings, $\overline{M} - M(C)$, as well as the transfers to the intermediary, $\alpha_I(C) = \alpha(C) + Y(C)$, we have

$$\frac{P(\overline{C})}{1+g} g = \overline{M} - M(C) + \alpha(C) + Y(C) \iff g(C) = \frac{\overline{C} - C + \alpha(C)}{P(\overline{C}) - \left[\overline{C} - C + \alpha(C)\right]} \tag{55}$$

where we used the fact that $\overline{M} = \overline{C}$ and the definition $C = M - Y$. A negative denominator would imply that there is a violation of LC. With LC and its implied constraint on α, (21), the denominator is non-negative (strictly so unless $C = \underline{C} < 0$), and we can always implement the optimal allocation via an LC-consistent common equity issuance.

Regardless of if LC is imposed, the post issuance price is given by

$$P_E(C) = P(\overline{C}) - \left[\overline{C} - C + \alpha(C)\right] \tag{56}$$

Note that $P_E'(C) = 1 - \alpha'(C) \geq 0$, with strict inequality for $C < \overline{C}$. Further, as LC is binding, the post issuance price corresponds to the pre-issuance price, so that

$$P_E(C) = P(C) \quad \text{and} \quad g(C) = \frac{P(\overline{C}) - P(C)}{P(C)}. \tag{57}$$

For $C \to \underline{C}$, existing shareholders are completely diluted, i.e., $\lim_{C \to \underline{C}} g(C) = \infty$, while the cash amount raised stays finite. In our restricted equity implementation, the firm itself only issues $g(C) - g_E(C)$ new shares, while the intermediary sells its $g_E(C)$ vesting shares.

[22]Without the normalization, g can be interpreted as the required growth rate in outstanding shares.

3.2.5 Discussion: An Overlapping Pecking-order

In our implementation, for $C > 0$, the credit line is used *simultaneously* with cash, with the credit line usage being indexed by $\beta(C)$. Thus, at $C = \overline{C}$ we have $\beta(\overline{C}) = 0$, so the credit line is not used, while its usage $\beta(C)$ increases as C declines. Next, restricted equity is not used until all cash reserves have been exhausted, i.e., until the point $M = Y = 0 \iff C = 0$. For any $C < 0$, financing occurs via a credit line and the sale/ buyback of restricted equity shares to/from the intermediary according to the schedule $g_E(C)$. Thus, an overlapping pecking order of financial instruments arises, with cash being used with highest intensity first, then the credit line, and finally restricted equity. Unlike in the traditional pecking order theory of Jensen and Meckling (1976), here the pecking order is in terms of intensity of use, and not in terms of a bang-bang solution: the firm always simultaneously uses two instruments — with the credit line always being present — to manage its liquidity, but uses those instruments to different degrees.

4 Discussion and Extensions

In this section, we provide further discussion of the results of the model through the lens of our implementation. First, we discuss which firms face the risk of future liquidations, and how financial market access impacts this risk. We interpret the optimal contract's equity stake as a PE or Growth Equity stake, and show how the level of PE activity is related to financial market access. Second, we investigate the value of intermediation, i.e., the value of costly on-demand financing. We do this by pitting our full model against two benchmarks: the no-intermediary-financing contract and the no-promises contract. Next, we provide an extension to one key assumption of the model — the exogenous Poisson access to financial markets – by endogenizing the intensity π via an optimal search extensions. Finally, we collect the empirical implications of the model.

4.1 Firm survival and Financial Market Development

Deferred payouts and firm survival in the absence of financial markets. Given that deferred payouts are available, why does the firm not reduce cash-holdings to zero in all states and simply use deferred payouts instead? For the remainder of this section, assume $\pi = 0$. Note that keeping cash at zero is equivalent to the intermediary absorbing all cash-flow shocks, which we termed autarky before. The issue is that promise keeping can only be fulfilled if the intermediary is actually willing to run the firm in autarky. First, consider the case in which the intermediary would not be willing to run the firm in autarky, i.e., $Y^A < 0$. Then, the firm optimally holds cash and uses the intermediary's funds to buffer some of its shocks dX. The firm's cash buffer allows it to reduce the reliance on intermediary's funds sufficiently so that the intermediary indeed is willing to provide such funds.

Next, consider the case in which the intermediary would be willing to run the firm in autarky, i.e. $Y^A > 0$. Now, it is possible to run the firm without cash, i.e., $M = 0$. But always doing so may not be optimal, as $\overline{C} > 0$ indicates. The reason is that the intermediary's funds are expensive due to the intermediary's risk-aversion: there is a trade-off between the cost of funds, expressed through the risk-premium $\frac{\rho r}{2}(\beta_t \sigma)^2$, and the cost of holding cash, in the form of the inefficiency λM_t. The optimal contract implements the firm holding some cash reserves after a good sequence of shocks, and running down these cash-reserves completely after a bad sequence of shocks and subsequently relying purely on the intermediary for cash-flow buffers. Nevertheless, this is different from selling the firm completely to the intermediary at the outset, as doing so would be inefficient: the outside investors are risk-neutral, so should bear most of the risk, even though transferring risk to them requires the use of the costly cash "machinery". Viewed differently, setting $\overline{C} = 0$ simply implies that $dI = dX$, the autarky solution. But we have shown that we can do better by holding some cash as it allows the risk-neutral investors to absorb some risk, leading to a value increase.

Refinancing and Financial Market Development. Next, let us consider the impact of better financial market access, i.e. increasing π. This is depicted in the left panel of Figure 4 which shows the upper (solid black) and lower boundaries (solid red) as a function of $1/\pi$, the expected time until market access. Our model shows that as π increases, the intermediary becomes more willing to take equity positions in firms even when $Y^A < 0$, as indicated by the red line. Effectively, the intermediary absorbs negative cash-flow shocks until the firm has access to financial markets again or saves itself out of its predicament via retained earnings. In other words, the intermediary acts like a PE/Growth capital fund: when $C < 0$ and negative shocks hit, it builds up an equity stake $Y > 0$ in return for some of the projects cash-flows (which would at most yield value Y^A to the intermediary absent refinancing) and a refinancing exit yielding a payout $\alpha_I(C) = \alpha(C) + Y$. This is efficient as it allows the intermediary to sell its equity stake to outside investors that due to risk-neutrality have a higher valuation of the firm, but due to frictions cannot invest continuously.

Next, the solid black line maps into firm net value at initiation via (30): a lower $\overline{C}$ translates into higher net value at initiation. Thus, the left panel of Figure 4 via its upward sloping solid

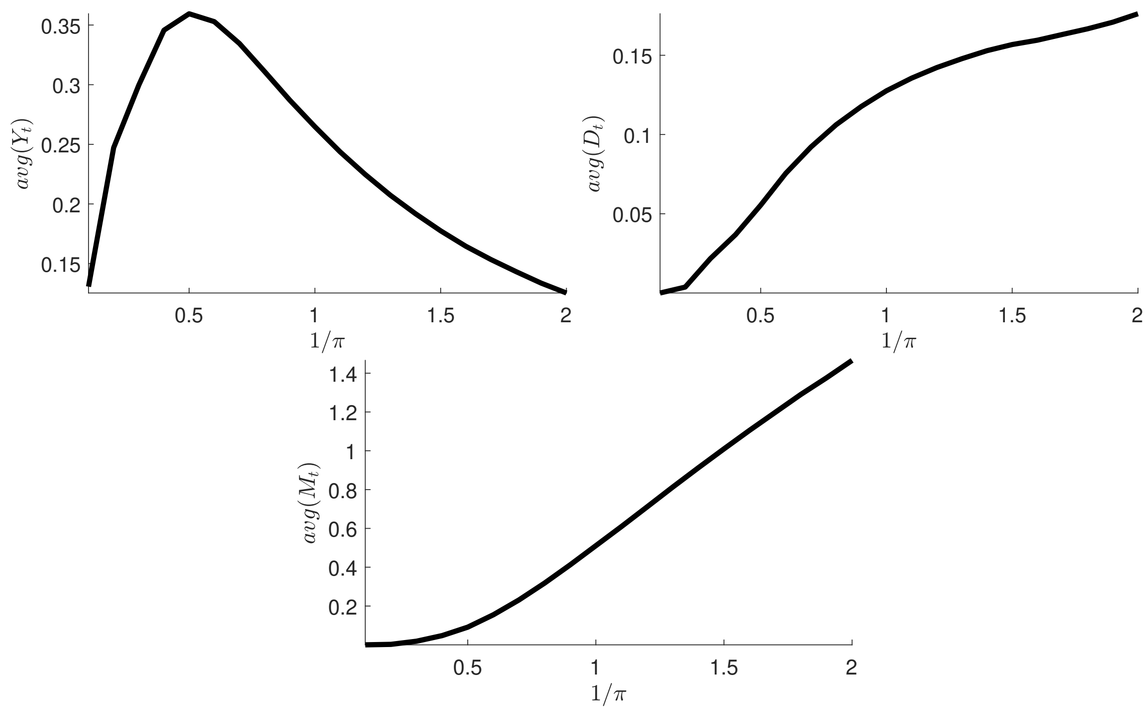

Figure 3: **Average levels of Y, D, M in steady state:** Top left panel depicts the average equity position in the economy, top right panel the average balance on the credit line, and the bottom panel the average cash-position in the economy, all as a function of the expected time until the next refinancing, $1/\pi$.

black line shows that as the time between refinancing opportunities shrinks, firm value increase. Further, we see a distinct kink in the solid black line when the firm transitions from survival to liquidation, i.e., when the red line hits its upper bound of zero.

Thus, as financial markets develop and π increases, our model implies that — all else equal — we should see a larger proportion of firms with $\underline{C} < 0$. In aggregate, Figure 3 shows how intermediary activity at first increases from zero as π rises from 0, but vanishes as $\pi \to \infty$ as intermediary stakes are immediately bought out even though the willingness to takes such stakes increases. Thus, the largest amount of intermediary activity occurs for intermediate π. We interpret the equity stake as a PE investment. In general, PE firms provide two services: first, they improve *operational* performance by changing management, while second they also provide temporary financing to the firm. Our model here excludes the first service, and purely concentrates on the second service. The intermediary's equity position arises purely to keep a valuable production process alive, i.e., to overcome financial constraints.

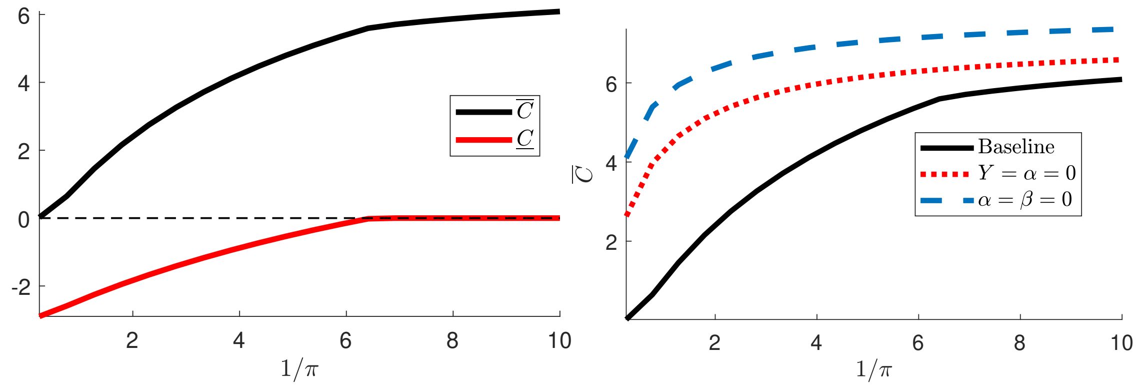

Figure 4: **Left panel**: Comparative statics of the boundaries $\underline{C}, \overline{C}$ w.r.t. to the expected time until refinancing, $1/\pi$. **Right panel**: Comparative statics of the boundaries $\underline{C}$ w.r.t. $1/\pi$ in the baseline case (black solid), the no-promises case (red dotted), and the no-intermediary-financing case (blue dashed).

4.2 The Value of Intermediation

In this subsection, we investigate the value of the optimal contract by contrasting it against two benchmarks: First, against the no-intermediary-financing benchmark, i.e., $\alpha = \beta = 0$, and second, against the no-promises benchmark, i.e., $Y = \alpha = 0$.

The value of on-demand financing. Consider restricting the contracting space to only no-intermediary-financing contracts $\alpha = \beta = 0$, which immediately implies $Y = 0$.[23] The problem then effectively becomes a classical "pure" liquidity management problem with $C = M$: Liquidation occurs at $\underline{C} = 0$, while payouts occur at some $\overline{C}_{\alpha=\beta=0}$. The total value relation at the optimal payout boundary (30) still holds, and thus comparing value creation at initiation simply reduces to comparing optimal $\overline{C}$'s. As we artificially restricted the financing space, it must be that $\overline{C}_{\alpha=\beta=0} > \overline{C}$. The right-panel in Figure 4 illustrates $\overline{C}$: The no-intermediary-financing contract is the dashed blue line, while the fully optimal contract is the solid black line. The horizontal axis is given by $1/\pi$, which is the expected time until the next refinancing opportunity. We see that there is significant value gains from inside financing, as indicated by the gap between the two lines.

The value of credible promises. Let $\overline{Y}\left(\overline{C}\right) \equiv -\underline{C}\left(\overline{C}\right)$ be the maximum level of *credible* promises to the intermediary by the firm, something we will term financing capacity. Here, credible refers to promises that can fulfill the promise-keeping constraint (14). As Lemma 1 established, the

[23]This is optimal for an infinitely risk-averse intermediary, i.e., $\rho \to \infty$.

maximum level of credible promises $\overline{Y}$ increases with π, with $\lim_{\pi\to\infty}\overline{Y}=\frac{\mu}{r}$. In other words, better access to markets relaxes the contracting problem by allowing higher levels of credible promises, which in turn improves the contract.

Next, let us investigate the value of such extended promises by comparing the model to a no-promises contract, i.e., $Y=\alpha=0$ always. This does not rule out survival at $\underline{C}=0$, but by (32), we see that with $\beta(0)=1$ we can have a (slow) reflection in that $\mu_C(0)>0$ but $\sigma_C(0)=0$. Again, we compare values at initiation, i.e., at $C=\overline{C}$. The right-panel in Figure 4 illustrates: The no-promises contract is the dotted red line, while the fully optimal contract is the solid black line. The horizontal axis is given by $1/\pi$, which is the expected time until the next refinancing opportunity. The gap between the lines indicates that there are significant value gains from relying on (credible) promises. Further, we note that the no-promises contract lies between the no-intermediary-financing and the fully optimal case, something that is natural as the contract with no-intermediary-financing is a subset of all no-promises contracts.

4.3 Extension: Searching for capital

To show robustness of the results to the modeling assumption of the financial friction as a Poisson arrival, let us allow firms to actively search for financing, i.e., choose π, at a monetary search cost $c(\pi)$. We will consider two specifications. In either case, we observe a negative relationship between refinancing and the current net-cash position C once LC is imposed.

Convex search cost. First, consider convex costs $c(\pi)$ with $c'(\pi)>0$ as well as $c''(\pi)>0$ and $c(0)=0$ to find outside intermediaries. As this cost has to be paid out of the firm's cash-holdings, we have

$$rP(C)=\max_{\substack{\pi\geq 0,Y\geq\max\{0,-C\}\\ \beta,C^*,\alpha\in\mathcal{S}(C^*)}}\Big\{P'(C)\Big[\mu+(r-\lambda)C-\lambda Y-\frac{\rho r}{2}(\beta\sigma)^2-c(\pi)\Big]+P''(C)\frac{\sigma^2}{2}(1-\beta)^2 + \pi\left(P'(C)\frac{1-e^{-\rho r\alpha}}{\rho r}+\left[P(C^*)-P(C)-(C^*-C+\alpha)\right]\right)\Big\} \tag{58}$$

Note that π did not enter any of the optimal policies (23)-(27), and thus they remain unchanged. The optimal search intensity $\pi(C)$ is determined by the FOC

$$c'(\pi(C)) = \underbrace{\left[\frac{1-e^{-\rho r \alpha(C)}}{\rho r} + \frac{P\left(\overline{C}\right) - P(C) - \left[\overline{C} - C + \alpha(C)\right]}{P'(C)}\right]}_{\equiv G(C)} \tag{59}$$

where the term in square brackets is the marginal gain of additional search intensity, $G(C)$. Note that $G(\overline{C}) = 0$. By the convexity of $c(\cdot)$, if the gain from refinancing is larger for small C, i.e., $G'(C) < 0$, then firms search with higher intensity in lower liquidity states, i.e., $\pi'(C) < 0$. This always holds in our baseline case of LC as α is constrained and the gain simplifies to $G(C) = \frac{1-e^{-\rho r \alpha(C)}}{\rho r}$. Thus, under LC, poorly capitalized firms search at a higher intensity than well capitalized firm, and as before raise more money upon refinancing.

Linear search cost. Next, consider a linear search cost $c(\pi) = \pi \cdot k_\pi$ for some exogenous constant $k_\pi > 0$ and some exogenous upper limit on π of $\overline{\pi}$. Then, a bang-bang situation ensues and the firm searches for capital at intensity $\pi = \overline{\pi}$ if and only if

$$k_\pi \leq G(C) \tag{60}$$

and otherwise does not search, i.e., $\pi = 0$. Again, if the gain is smaller for lower C, as it is under LC, we will observe poorly capitalized firms search for capital and refinance via equity markets if possible, while well-capitalized firms abstain from costly search and thus do not refinance.

4.4 Summary of Empirical Implications

Besides the implications of the implementation that we have previously discussed, the model predicts the following:

- Firms raise more money the more poorly capitalized they are upon refinancing, as $C^* - C$ is decreasing in C.
- Fixing the production process, a firm's net cash position becomes less volatility for firms with low net cash positions, as $\beta'(C) < 0$.
- More liquid capital markets (higher π) result in more firm survival and more equity stakes and exits by intermediaries.

- Total intermediation activity (positions appropriately weighted by the time they take to unwind) is a hump-shaped function of π, reflecting the fact that as capital markets become very accessible, intermediaries effectively take smaller and smaller positions before exiting. Figure 3 illustrates.

- As markets become more liquid, firms are using the credit-line to a lesser degree (although the total credit line limit decreases as π increase).

- Average cash-holdings in the economy shrink as π increases. With better access to market, the firm's need to (inefficiently) hedge liquidity shocks via cash decreases.

- More liquid capital markets (higher π) lead to a lower payout boundary for financially constrained firms, i.e., $\frac{\partial \overline{C}}{\partial \pi} < 0$.

5 Conclusion

We consider a firm that has infrequent access to public capital markets and an internal cost of cash. In the interim, it contracts with a deep-pocketed but costly intermediary subject to limited commitment. Under the optimal financing agreement, the intermediary assumes a fraction of cash-flow risk that decreases in the firm's net-cash position. A credit-line with a state-dependent interest rate and early-repayment incentives, restricted equity that vests upon market access, and cash reserves jointly implement the optimal contract. Initially, the firm uses cash and the credit-line simultaneously. If the firm runs out of cash, it either liquidates or uses equity and the credit-line to sustain operation. Thus, an overlapping pecking order arises. Upon market access, the firm issues common equity to retire the credit line. Although the intermediary has deep pockets, it allows riskier firms to face endogenous liquidation. To ensure the survival of less risky firms, it acquires restricted-equity that it sells upon market access. The set of firms facing liquidation is smaller if market access, and therefore intermediary exit, is more likely. Importantly, the optimal financing agreement is renegotiation proof.

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A Proofs:

Proof $\beta'(C) < 0$ under LC. Assume LC, which we later show is always binding. Then, the simplified HJB equation (31) is further simplified to

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right)\right]. \tag{61}$$

As $P(C) \geq 0$ and $P'(C) \geq 1$, it must be that

$$\hat{\mu}_C(C) \equiv \left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right)\right] \geq 0. \tag{62}$$

Differentiate both sides with respect to C, so that

$$\lambda \mathbf{1}_{\{C \geq 0\}} P'(C) = P''(C)\hat{\mu}_C(C) + P'(C)\left[-\frac{\rho r}{2}\beta'(C)\sigma^2 + \pi e^{-\rho r \alpha(C)}\alpha'(C)\right]. \tag{63}$$

That is,

$$\lambda \mathbf{1}_{\{C \geq 0\}} P'(C) - P''(C)\hat{\mu}_C(C) - \pi e^{-\rho r \alpha(C)}\alpha'(C) = -P'(C)\frac{\rho r}{2}\beta'(C)\sigma^2. \tag{64}$$

As $\alpha'(C) < 0$ and by the concavity of $P(C)$, the left-hand-side is strictly positive. Thus, $\beta'(C) < 0$.

Concavity of value function. Define the jump in the value function upon refinancing as

$$J(C) = P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha\right) \tag{65}$$

so that

$$J'(C) = 1 - P'(C) - \alpha'(C). \tag{66}$$

and write the HJB equation as

$$rP(C) = \max_{\beta \in [0,1]}\left\{P'(C)\mu_C(C) + \frac{P''(C)}{2}\sigma^2(1 - \beta(C))^2 + \pi J(C)\right\} \tag{67}$$

Differentiate the HJB equation to obtain

$$P'''(C) = \frac{2}{(1-\beta)^2\sigma^2}\left(P'(C)\lambda \mathbf{1}_{\{C \geq 0\}} - P''(C)\mu_C(C) - \pi\left(e^{-\rho r \alpha(C)}P'(C)\alpha'(C) + J'(C)\right)\right). \tag{68}$$

When $C = \overline{C}$, then $\alpha(C) = 0$ under either scenario. Thus, $P'''(C) > 0$ in a neighbourhood of $\overline{C}$, so that $P''(C) < 0$ in a neighbourhood of $\overline{C}$. Define $\hat{C} = \sup\{C \geq 0 : P''(C) \geq 0\}$ and suppose to the contrary that $\hat{C} < \overline{C}$. As $P''(C) < 0$ in a neighbourhood of $\overline{C}$, it follows by continuity that $P''(\hat{C}) = 0$.

We show that $P'''(\hat{C}) > 0$ and, to do so, we distinguish between three different cases. First, suppose that LC is binding so that

$$\alpha(C) = P(\overline{C}) - \overline{C} + C - P(C), \tag{69}$$

so that $\alpha'(C) = 1 - P'(C) < 0$, and $J(C) = J'(C) = 0$. Then, $P'''(\hat{C}) > 0$. Second, suppose that LC is not binding, so that

$$\alpha(C) = \frac{\log P'(C)}{\rho r}. \tag{70}$$

Then,

$$e^{-\rho r \alpha(C)} P'(C)\alpha'(C) + J'(C) = 1 - P'(C) < 0. \tag{71}$$

Thus, $P'''(\hat{C}) > 0$. Thus, $P'''(\hat{C}) > 0$. Due to $P'''(\hat{C}) > 0$, there exists $C' > \hat{C}$ so that $P''(C') > 0$, which contradicts the definition of $\hat{C}$. Therefore, $\hat{C} = \overline{C}$ and $P''(C)$ for all $C < \overline{C}$.

Weak comparative statics of $\overline{C}$ w.r.t. π. First, it is straightforward to show that $\frac{\partial \overline{C}}{\partial \pi} \leq 1$. Fix a π, and consider an increase to $\pi' > \pi$. Then, there always exists the following probabilistic policy: follow the previous, i.e, π, refinancing policy with probability $q = \frac{\pi}{\pi'}$ upon arrival of an opportunity, and with probability $1 - q$ ignore the opportunity when it arrives. But this leads to an effective refinancing rate of π. Thus, given that this policy is available, it must be that the firm is at least as well of with the higher π'. By $P\left(\overline{C}\right) = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$, we therefore must have $\overline{C}' \leq \overline{C}$.

Comparative Statics of Payout Boundary $\overline{C}$. We first establish some auxiliary results and then prove all claims of the Corollary separately. Here, η is an arbitrary model parameter and define $\partial_\eta(\cdot) \equiv \frac{\partial(\cdot)}{\partial \eta}$. We state the following auxiliary lemma:

Lemma 2. *Define*

$$\hat{\mu}_C(C) = \mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right) C - \frac{\rho r}{2}\beta(C)^2\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right) \tag{72}$$

and

$$\sigma_C(C) = \sigma(1 - \beta(C)). \tag{73}$$

Recall that $\alpha(C) = P\left(\overline{C}\right) - \overline{C} - P(C) - C$*. For* $\tau = \inf\{t \geq 0 : C_t = \underline{C}\}$ *and* $dC_t = \mu_{Ct}dt + \sigma_{Ct}dZ_t - dDiv_t$ *the following holds:*

$$\frac{dP(C)}{d\eta} = \frac{\partial P(C)}{\partial \eta} = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(P'(C_t)\partial_\eta \hat{\mu}_C(C_t) + \frac{P''(C_t)}{2}\partial_\eta \sigma_C(C_t)^2 - P(C_t)\partial_\eta r\right)dt\middle|C_0 = C\right]. \tag{74}$$

Proof. Let η a model parameter and β, Y the optimal controls in optimum. Let $P_\eta(C) = \partial_\eta P(C)$, and $P_{\eta\eta}(C) = \partial_{\eta\eta}P(C)$. Define

$$J(C) = \left[P\left(\overline{C}\right) - P(C) - \left(\overline{C} - C + \alpha\right)\right] \tag{75}$$

and note that in optimum $J(C) = 0$ everywhere, so that $\partial_\eta J(C) = 0$ for all C. We can write the HJB equation (under the optimal controls) for any $C \in [0, \overline{C}]$ as

$$rP(C) = \hat{\mu}_C(C)P'(C) + \frac{P''(C)}{2}\sigma_C(C)^2 \tag{76}$$

where we used that $J(C) = 0$. Next, we differentiate this HJB equation w.r.t. η to obtain

$$P(C)\partial_\eta r = -\, rP_\eta(C) + P'(C)\partial_\eta \mu_C(C) + \frac{P''(C)}{2}\partial_\eta \sigma_C(C)^2 + \mu_C(C)P'_\eta(C) + \sigma_C(C)^2 P''_\eta(C) \tag{77}$$

where $\partial_\beta P(C) = 0$ by the envelope theorem (and $J(C) \equiv J_\eta(C) = 0$). The boundary conditions are $P'_\eta(\overline{C}) = P''_\eta(\overline{C}) = 0$. Note that provided smoothnesss, we can interchange the order of differentiation, such that:

$$P'_\eta(C) \equiv \frac{\partial}{\partial \eta}\frac{\partial P(C)}{\partial C} = \frac{\partial}{\partial C}\frac{\partial P(C)}{\partial \eta} \text{ and } P''_\eta(C) \equiv \frac{\partial}{\partial \eta}\frac{\partial^2 P(C)}{\partial C^2} = \frac{\partial^2}{\partial C^2}\frac{\partial P(C)}{\partial \eta}. \tag{78}$$

We can solve for:

$$rP_\eta(C) = \left(P'(C)\partial_\eta \hat{\mu}_C(C) + \frac{P''(C)}{2}\partial_\eta \sigma(C)_C^2 - P(C)\partial_\eta r\right) + \frac{\mathbb{E}dP_\eta(C)}{dt}. \tag{79}$$

Invoking the Feynman-Kac formula yields the desired integral expression for $P_\eta(C)$. □

Using the above Lemma, we obtain

$$\frac{dP(C)}{d\pi} = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(P'(C_t)\left(\frac{1-e^{-\rho r\alpha(C_t)}}{\rho r}\right)\right)dt\middle|C_0 = C\right] > 0 \tag{80}$$

$$\frac{dP(C)}{d\rho} = \mathbb{E}\left[\int_0^\tau e^{-rt}P'(C)\left(\pi\frac{\partial}{\partial\rho}\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right) - \frac{r}{2}(\beta(C_t)^2\sigma^2\right)dt\middle|C_0 = C\right] < 0. \tag{81}$$

In addition, we evaluate the HJB at the boundary at $C = \overline{C}$, i.e.,

$$rP(\overline{C}) = \mu + (r-\lambda)\overline{C}. \tag{82}$$

We differentiate the above relation with respect to parameter $\eta \notin \{\mu, r, \lambda\}$ to obtain

$$rP_\eta(\overline{C}) + r\partial\eta\overline{C} = (r-\lambda)\partial\eta\overline{C} \quad \Longleftrightarrow \quad \partial_\eta\overline{C} = -\frac{r}{\lambda}P_\eta(\overline{C}). \tag{83}$$

Thus, $\frac{\partial C}{\partial\pi} < 0$ and $\frac{\partial C}{\partial\pi} > 0$.

Optimality of $\underline{C}$ so that $\mu_C(\underline{C}) = 0$. The proof proceeds in two steps: first, we show that the smallest feasible lower boundary $\underline{C}$ consistent with limited liability, which we call $\underline{C}_0$, must have $P(\underline{C}_0) = \mu_C(\underline{C}_0) = 0$. Second, we highlight that any lower boundary $\underline{C} > \underline{C}_0$ is not imposing a different boundary condition, but is rather the outcome of a choice of β. In other words, the boundary condition on the HJB comes from $\underline{C}_0$. The maximization over β then contains a maximization over possible $\underline{C}$ as the contract could pick $\beta(C) = 1$ for $C \in (\underline{C}_0, \underline{C})$ and thus introduce the effective lower boundary $\underline{C} > \underline{C}_0$, but does not do so.

We define $\underline{C}_0 = \underline{C}^{\text{Survival}}$ as the lowest value such that the payout agreement can ensure $C \geq \underline{C}$ with certainty without violating limited liability. Any such boundary $\underline{C}$ must have that $\sigma_C(\underline{C}) = 1$ and $\mu_C(\underline{C}) \geq 0$. The HJB equation under the optimal controls $\alpha(C)$ and $\beta(C)$ can be rewritten as

$$rP(C) = P'(C)\mu_C(C) + \frac{P''(C)(\sigma_C(C))^2}{2} + \pi J(C), \tag{84}$$

with jump in the value function upon refinancing

$$J(C) \equiv P(\overline{C}) - \overline{C} - P(C) + C - \alpha(C). \tag{85}$$

We will now consider the three possible cases:

1. Consider the *limited liability* scenario and assume it is binding, so that $J(C) = 0$. Then, $P(\underline{C}_0) > 0 \iff \mu_C(\underline{C}_0) > 0$. If $\mu_C(\underline{C}_0) > 0$, there exists $C' < \underline{C}_0$ with $P(C') > 0 \iff \mu_C(C') > 0$, which contradicts the definition of $\underline{C}_0$. Thus, $P(\underline{C}_0) = \mu_C(\underline{C}_0) = 0$.

2. Consider the *limited liability with commitment* scenario and suppose that the limited liability constraint binds so that $J(C) = -P(C)$. After rearranging the HJB, we note that this is analagous to the limited liability with commitment binding scenario above, except the effective discount rate changed from r to $r + \pi$.

3. Consider the *limited liability with commitment* scenario and suppose that the limited liability constraint does not bind, so that $\alpha(C)$ is unconstrained and $J(C) > -P(C)$. Suppose to the contrary that $P(\underline{C}_0) > 0$. Then, there exists $C' < \underline{C}_0$ with $\alpha(C') \geq \alpha(\underline{C}_0)$ and $\mu_C(C') \geq 0$, a contradiction. Thus, $P(\underline{C}_0) = 0$. Suppose to the contrary that $\mu_C(\underline{C}_0) > 0$. As $P(\underline{C}_0) = 0$, $J(\underline{C}_0) < 0$. But, $J(\underline{C}) \geq -P(\underline{C}) = 0$, a contradiction. Thus, $P(\underline{C}_0) = \mu_C(\underline{C}_0) = 0$.

We thus conclude that $P(\underline{C}_0) = \mu_C(\underline{C}_0) = 0$.

Next, consider the optimization (22). Note that $\underline{C}$ is not a separate choice variable from the other policies as the main text may imply. Rather, the optimization is picking β subject to the constraint that promises have to be credible. In other words, we are maximizing w.r.t. β with the constraint being that at the furthest feasible lower bound $\underline{C}_0$ we require $\beta(\underline{C}_0) = 1$ and $P(\underline{C}_0) = 0$. That is, any arbitrary lower bound $\underline{C} \in (\underline{C}_0, 0)$ can be implemented by simply setting $\beta(C) = 1$ on $C \in (\underline{C}_0, \underline{C})$. However, we see that the optimal $\beta(C)$, given the concavity of $P(C)$, is defined by (27) regardless of the lower boundary. Because $P''(\underline{C}) \neq -\infty$ for any $\underline{C} > \underline{C}_0$ by the HJB, it must be that $\beta(C) < 1$ for all $C \in \left(\underline{C}_0, \overline{C}\right)$. Thus, the optimal lower boundary is given by $\underline{C}_0$.

Smallest viable $\underline{C} < 0$ under full commitment. Let $\underline{C}^*$ denote the smallest solution $\underline{C}$ to $\mu_C\left(\underline{C}\right) \geq 0$. It is related to the full commitment case, as it arises when we pick $\alpha\left(\underline{C}\right) = \infty$, which results in

$$\underline{C}^* = -\left(\frac{\mu}{r} - \frac{\rho}{2}\sigma^2 + \frac{\pi}{\rho r^2}\right) = -\left(Y^{\mathrm{A}} + \frac{\pi}{\rho r^2}\right). \tag{86}$$

Note that this solution is inconsistent with $P\left(\underline{C}\right) = 0$. Further, note that this implies that with full commitment, as $\pi \to \infty$, the firm can make unbounded promises, i.e., $\underline{C}^* \to -\infty$.

Maximum credible promise / financing capacity $\overline{Y}=-\underline{C}$. Define

$$z(\pi)\equiv\frac{\pi}{r}\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C}+\frac{\pi}{\rho r^2}-\frac{\rho}{2}\sigma^2\right]\right\}\geq 0 \tag{87}$$

and note that $w(z)>0 \iff z>0$. Define $f(\pi)\equiv w(z(\pi))-\frac{\pi}{r}$, so that for a given $\overline{C}$, (35) can be expressed as

$$\underline{C}^{\text{survival}}(\pi)=\frac{f(\pi)}{\rho r}-Y^A. \tag{88}$$

Then $f'(\pi)<0$ implies larger credible promises $\frac{d}{d\pi}\overline{Y}>0$, and vice-versa. Differentiating w.r.t. π, we have

$$\begin{aligned}
f'(\pi) &= w'(z(\pi))\,z'(\pi)-\frac{1}{r}\\
&=\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C}+\frac{\pi}{\rho r^2}-\frac{\rho}{2}\sigma^2\right]\right\}\frac{1}{r}w'(z)\left(1+\frac{\pi}{r}\right)-\frac{1}{r}\\
&=\frac{1}{\pi}\left[z\cdot w'(z)\left(1+\frac{\pi}{r}\right)-\frac{\pi}{r}\right]\\
&=\frac{1}{\pi}\left[w(z)\frac{1+\frac{\pi}{r}}{1+w(z)}-\frac{\pi}{r}\right]\\
&=\frac{1}{\pi}\frac{1}{1+w(z)}\left[w(z)\left(1+\frac{\pi}{r}\right)-\frac{\pi}{r}(1+w(z))\right]\\
&=\frac{1}{\pi}\frac{1}{1+w(z)}\left[w(z)-\frac{\pi}{r}\right]\\
&=\frac{1}{\pi}\frac{1}{1+w(z)}f(\pi)
\end{aligned} \tag{89}$$

where we used the Lambert-w relation $w'(z)=\frac{w(z)}{z(1+w(z))}$. Using the fact that $w'(0)=1$, we have

$$f'(0)=\frac{1}{r}\left[\exp\left\{\rho r\left[\frac{\lambda}{r}\overline{C}-\frac{\rho}{2}\sigma^2\right]\right\}-1\right]. \tag{90}$$

By (30), we have $\frac{\lambda}{r}\overline{C}<\frac{\rho}{2}\sigma^2$ which implies $f'(0)<0$, and in turn implies both $f(\pi)<0$ and $f'(\pi)<0$. Next, note that for $\pi>0$ we also have

$$\frac{\partial f(\pi)}{\partial\overline{C}}=w'(z(\pi))\,\lambda\rho z(\pi)=\lambda\rho\frac{w(z)}{1+w(z)}>0, \tag{91}$$

so if $\overline{C}$ shrinks, the maximum credible promise $\overline{Y}$ increases.

Next, let us calculate $\lim_{\pi\to\infty}\underline{C}\left(\overline{C}(\pi)\right)$ where $\overline{C}(\pi)$ is the *continuously re-optimized* payout boundary. To this end, we will first investigate the derivative $\underline{C}'\left(\overline{C}\right)$. We have

$$\underline{C}'\left(\overline{C}\right) = \frac{\frac{\partial f(\pi)}{\partial \overline{C}}}{\rho r} = \frac{\lambda}{r}\frac{w\left(z\right)}{1+w\left(z\right)} \leq \frac{\lambda}{r}. \tag{92}$$

Next, let us fix $\overline{C}$ and calculate the limit of $\underline{C}\left(\overline{C}\right)$ as $\pi \to \infty$. We will use the asymptotic expansion $w\left(z\right) = \log z - \log\left(\log z\right) + o\left(1\right)$ for $z \to \infty$. Consequently,

$$\begin{aligned}
\lim_{\pi\to\infty} f\left(\pi\right) &= \lim_{\pi\to\infty} w\left(z\left(\pi\right)\right) - \frac{\pi}{r} \\
&\approx \lim_{\pi\to\infty}\left\{\log\frac{\pi}{r} + \rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right] - \log\left(\log\frac{\pi}{r} + \rho r\left[\frac{\lambda}{r}\overline{C} + \frac{\pi}{\rho r^2} - \frac{\rho}{2}\sigma^2\right]\right) - \frac{\pi}{r}\right\} \\
&= \lim_{\pi\to\infty}\left\{\log\frac{\pi}{r} + \rho\lambda\overline{C} - r\frac{\rho^2}{2}\sigma^2 - \log\left(\log\frac{\pi}{r} + \rho\lambda\overline{C} - r\frac{\rho^2}{2}\sigma^2 + \frac{\pi}{r}\right)\right\} \\
&= \rho r\left[\frac{\lambda}{r}\overline{C} - \frac{\rho}{2}\sigma^2\right]
\end{aligned} \tag{93}$$

as $\lim_{x\to\infty}\left[\log x - \log\left(x + \log x - a\right)\right] = 0$ for any bounded a. Then, we have

$$\lim_{\pi\to\infty} \underline{C}\left(\overline{C}\right) = -\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C}\right]. \tag{94}$$

As $\overline{C}\left(\pi\right)$ is bounded above and below by parameters independent of π, its maximum change in response to π is bounded. When $\pi \to \infty$ we have that $\overline{C} \to 0$ (this can be easily proved via the payout boundary of the no risk-sharing contract $\alpha = \beta = 0$, which is an upper bound), as the market is becoming fully liquid. This then implies that $\underline{C}$ converges to the risk-neutral NPV of the project.

Optimal contract without promises. Let us constrain the contracting space to only no-promises contracts, i.e., $Y = 0$ always. Then a (slow) reflection can occur at $\underline{C} = 0$. In this case, we must have $\beta\left(0\right) = 1$, which is only consistent with survival if and only if

$$\mu_C\left(0\right) = \mu - \frac{\rho r}{2}\sigma^2 + \pi\frac{1 - e^{-\rho r\alpha(0)}}{\rho r} \geq 0. \tag{95}$$

Evaluating the HJB (31) at $C = 0$ and imposing $\beta\left(0\right) = 1$, we have

$$rP\left(0\right) = P'\left(0\right)\left[\mu - \frac{\rho r}{2}\sigma^2 + \pi\frac{1 - e^{-\rho r\alpha(0)}}{\rho r}\right] + \pi\left[P\left(\overline{C}\right) - \overline{C} - \alpha\left(0\right) - P\left(0\right)\right]. \tag{96}$$

The boundary condition at $C = 0$ will depend on which $\alpha(0)$ the contract stipulates.

- Suppose α is unconstrained at 0, i.e., $\alpha(0) = \alpha_U(C) = \frac{\log P'(0)}{\rho r}$. Then, we have

$$\begin{aligned}(r+\pi)P(0) =& P'(0)\left[\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(0)}\right)\right] + \pi\left[P\left(\overline{C}\right) - \overline{C} - \frac{\log P'(0)}{\rho r}\right] \\ =& P'(0)\left[\mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\right] + \pi\left[\frac{\mu}{r} - \frac{1}{\rho r} - \frac{\lambda}{r}\overline{C} - \frac{\log P'(0)}{\rho r}\right] \end{aligned} \tag{97}$$

  which yields a boundary condition that connects $P(0)$ in a non-linear fashion with $P'(0)$ and $\log P'(0)$. The $\mu_C(0) \geq 0$ restriction can be simplified to

$$P'(0) \geq \frac{\frac{\pi}{\rho r}}{rY^A + \frac{\pi}{\rho r}}. \tag{98}$$

  Using the Lambert-w function, we can express $P'(0)$ as a function of $P(0)$

$$P'(0) = \frac{-\frac{\pi}{\rho r}\cdot w\left(\frac{\left[rY^A + \frac{\pi}{\rho r}\right]}{-\frac{\pi}{\rho r}}\exp\left\{\frac{(r+\pi)P(0) - \pi\left[\frac{\mu}{r} - \frac{1}{\rho r} - \frac{\lambda}{r}\overline{C}\right]}{-\frac{\pi}{\rho r}}\right\}\right)}{\left[rY^A + \frac{\pi}{\rho r}\right]} \tag{99}$$

  so that the restriction becomes

$$-w\left(\frac{\left[rY^A + \frac{\pi}{\rho r}\right]}{-\frac{\pi}{\rho r}}\exp\left\{\frac{(r+\pi)P(0) - \pi\left[\frac{\mu}{r} - \frac{1}{\rho r} - \frac{\lambda}{r}\overline{C}\right]}{-\frac{\pi}{\rho r}}\right\}\right) \geq 1. \tag{100}$$

  Multiplying through by $= -1$, we see that this requires $w(\hat{z}) \leq -1$, but the first branch of the Lambert-w function has a minimum of -1 at $\hat{z} = -\frac{1}{e}$. Thus, we can never have a reflection in which α is unconstrained.

- Suppose α is constrained at 0 by a limited liability with commitment constraint, i.e., $\alpha(0) = \alpha_{LCwC}(0) = P\left(\overline{C}\right) - \overline{C} = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C}$, so that

$$(r+\pi)P(0) = P'(0)\left[\mu - \frac{\rho r}{2}\sigma^2 + \pi\frac{1 - e^{-\rho r\left(\frac{\mu}{r} - \frac{\lambda}{r}\overline{C}\right)}}{\rho r}\right]. \tag{101}$$

  The $\mu_C(0) \geq 0$ restriction can be simplified to

$$Y^A + \frac{\pi}{r}\frac{1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C}\right]}}{\rho r} \geq 0. \tag{102}$$

- Suppose α is constrained at 0 by a limited liability without commitment constraint, i.e., $\alpha(0) = \alpha_{LC}(0) = P\left(\overline{C}\right) - \overline{C} - P(0) = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} - P(0)$ so that

$$rP(0) = P'(0)\left[\mu - \frac{\rho r}{2}\sigma^2 + \pi\frac{1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} - P(0)\right]}}{\rho r}\right] \tag{103}$$

which yields a boundary condition that connects $P(0)$ in a non-linear fashion with $P'(0)$ and $e^{\rho r P'(0)}$. The $\mu_C(0) \geq 0$ restriction can be simplified to

$$Y^A + \frac{\pi}{r}\frac{1 - e^{-\rho r\left[\frac{\mu}{r} - \frac{\lambda}{r}\overline{C} - P(0)\right]}}{\rho r} = \frac{P(0)}{P'(0)} \geq 0. \tag{104}$$

But by limited liability, we must have $P(0) \geq 0$, which we so far have not imposed. Notice that the constraint is the most relaxed when $P(0)$ is lowest (the LHS is decreasing in $P(0)$), and then coincides with the constraint of limited liability with commitment above.

Integral expression Y_t: Multiplying Y_t by e^{-rt} and applying Ito's lemma, we have

$$e^{rt}\cdot d\left(e^{-rt}Y_t\right) = \left(-rY_t dt + dY\right) = \left[\frac{\rho r}{2}\left(\beta_t\sigma\right)^2 + \pi\left(\alpha_t - \frac{1 - e^{-\rho r\alpha_t}}{\rho r}\right)\right]dt + \beta_t\sigma dZ_t + \alpha_t\left(d\Pi_t - \pi dt\right) - dI_t \tag{105}$$

Integrating from t to ∞, taking expectations and imposing the transversality condition, we have (14).

Total interest payment on Credit Line: Note that

$$\mu_I = \mu + \left[(r - \lambda)C - \mu_C\right]\mathbf{1}_{\{C \geq 0\}} \tag{106}$$

$$\mu_C = \mu + \left(r - \lambda\mathbf{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\left[1 - e^{-\rho r\alpha(C)}\right] \tag{107}$$

$$\mu_T = \left[\frac{\lambda}{\rho r} + \mu_C(C)\right]\mathbf{1}_{\{C \geq 0\}} - \left[\mu + \left(r - \lambda\mathbf{1}_{\{C \geq 0\}}\right)C\right] - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right]\left[1 - \alpha'(C)\right] \tag{108}$$

where we used

$$
\begin{aligned}
\mu_T(C) =& T'(C)\,\mu_C(C) + T''(C)\,\frac{\sigma_C^2(C)}{2} \\
=& T'(C)\,\mu_C(C) - \frac{\beta'(C)}{[1-\beta(C)]^2}\frac{\sigma^2[1-\beta(C)]^2}{2} \\
=& T'(C)\,\mu_C(C) - \frac{\sigma^2}{2}\beta'(C) \\
=& -\left[\frac{\rho r}{2}\beta^2(C)\,\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r}\right] \\
=& \mu_C\mathbf{1}_{\{C\geq 0\}}(C) - \left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r}\right] \\
& - \frac{\pi}{\rho r}\left\{\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \left(1 - \frac{1}{P'(C)}\right) + \left(1 - e^{-\rho r\alpha(C)}\right)\right\} \\
=& \left[\frac{\lambda}{\rho r} + \mu_C(C)\right]\mathbf{1}_{\{C\geq 0\}} - \left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C\right] - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right]\left[1 - \alpha'(C)\right]
\end{aligned}
\tag{109}
$$

where we used $T''(C) = -\frac{\beta'(C)}{[1-\beta(C)]^2}$ and (116) to substitute in for $\frac{\sigma^2}{2}\beta'(C)$. Plugging in for $\mu_I(C)$, we have

$$
\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbf{1}_{\{C\geq 0\}} + rY(C) - \frac{\pi}{\rho r}\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right]\left[1 - \alpha'(C)\right] \tag{110}
$$

For the unconstrained case $\alpha(C) = \frac{\log P'(C)}{\rho r}$, see (**??**), we have $\left[\frac{1}{P'(C)} - e^{-\rho r\alpha(C)}\right] = 0$. XXX For the constrained case of limited liability with commitment $\alpha(C) = P(C^*) - (C^* - C)$, see (**??**), we have $[1 - \alpha'(C)] = 0$. Both result in

$$
\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbf{1}_{\{C\geq 0\}} + rY(C) \tag{111}
$$

Lastly, consider the constrained case of limited liability without commitment $\alpha(C) = [P(C^*) - C^*] - [P(C) - C]$, see (**??**), we have $[1 - \alpha'(C)] = P'(C)$, and thus

$$
\mu_T(C) + \mu_I(C) = \frac{\lambda}{\rho r}\mathbf{1}_{\{C\geq 0\}} + rY(C) - \frac{\pi}{\rho r}\left[1 - P'(C)\,e^{-\rho r\alpha(C)}\right]. \tag{112}
$$

Risk-sharing slope $\beta'(C)$: To get $\beta'(C)$, we differentiate the HJB (31)

$$rP(C) = P'(C)\left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] + \pi\left[\frac{\mu-\lambda\overline{C}}{r} + C - \alpha(C) - P(C)\right] \tag{113}$$

to get

$$\begin{aligned} rP'(C) = & P''(C)\left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ & + P'(C)\left[\left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right) - \frac{\rho r}{2}\beta'(C)\sigma^2 + \pi\alpha'(C)e^{-\rho r\alpha(C)}\right] \\ & + \pi\left[1 - \alpha'(C) - P'(C)\right] \end{aligned} \tag{114}$$

Rearranging, we have

$$\begin{aligned} & \left[\pi\left(1 - \alpha'(C)e^{-\rho r\alpha(C)}\right) + \lambda\mathbf{1}_{\{C\geq 0\}} + \frac{\rho r}{2}\beta'(C)\sigma^2\right]P'(C) \\ & = P''(C)\left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] + \pi\left[1 - \alpha'(C)\right] \end{aligned} \tag{115}$$

Dividing through by $\rho r P'(C)$ and solving for $\frac{\sigma^2}{2}\beta'(C)$, we have

$$\begin{aligned} \frac{\sigma^2}{2}\beta'(C) = & \frac{P''(C)}{\rho r P'(C)}\left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ & + \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1 - \alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \\ = & -\frac{\beta(C)}{1-\beta(C)}\left[\mu + \left(r - \lambda\mathbf{1}_{\{C\geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right] \\ & + \frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)} - \frac{\pi}{\rho r}\left[1 - \alpha'(C)e^{-\rho r\alpha(C)}\right] - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \\ = & -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 + \frac{\pi}{\rho r}\alpha'(C)\left[e^{-\rho r\alpha(C)} - \frac{1}{P'(C)}\right] - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \end{aligned} \tag{116}$$

where we used $\frac{1}{\rho r}\frac{P''(C)}{P'(C)} = -\frac{\beta(C)}{1-\beta(C)}$. Note that regardless of which α applies, we have

$$\frac{\sigma^2}{2}\beta'\left(\overline{C}\right) = -\frac{\lambda}{\rho r}. \tag{117}$$

Further, note that for $\alpha(C) = \alpha_U(C) = \frac{\log P'(C)}{\rho r}$, we have

$$\frac{\sigma^2}{2}\beta_U'(C) = -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 - \frac{\pi}{\rho r}\left(1 - \frac{1}{P'(C)}\right) - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \tag{118}$$

and for $\alpha(C) = \alpha_{LC}(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]$, we have

$$\frac{\sigma^2}{2}\beta_{LC}'(C) = -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 - \frac{\pi}{\rho r}\left[P'(C) - 1\right]e^{-\rho r\alpha(C)} - \frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \tag{119}$$

while for $\alpha(C) = \alpha_{LCwC}(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] + C$, we have

$$\frac{\sigma^2}{2}\beta_{LCwC}'(C) = -\frac{\beta(C)}{1-\beta(C)}\mu_C(C) + \frac{\rho r}{2}\beta^2(C)\sigma^2 - \frac{\pi}{\rho r}\left[P'(C) - 1\right]e^{-\rho r\alpha(C)} - \frac{\pi + \lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r} \tag{120}$$

Limited liability always binding. Next, we will prove that limited liability is always binding, i.e., $\alpha_{LC}(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]$. Define the function

$$f(C) \equiv \alpha_U(C) - \alpha_{LC}(C) = \frac{\log P'(C)}{\rho r} - \left\{\left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]\right\} \tag{121}$$

$$f'(C) = \frac{P''(C)}{\rho r P'(C)} + P'(C) - 1 = -\frac{\beta(C)}{1-\beta(C)} + P'(C) - 1 \tag{122}$$

$$f''(C) = -\frac{\beta'(C)}{\left[1-\beta(C)\right]^2} + P''(C) = -\frac{1}{1-\beta(C)}\left[\frac{\beta'(C)}{1-\beta(C)} + \rho r P'(C)\beta(C)\right] \tag{123}$$

where we used $\frac{P''(C)}{\rho r P'(C)} = -\frac{\beta}{1-\beta}$. First, note that $f\left(\overline{C}\right) = f'\left(\overline{C}\right) = 0$, and from (117) above, we know that $\beta'\left(\overline{C}\right) = -\frac{2}{\sigma^2}\frac{\lambda}{\rho r}$ regardless of which α applies, so that $f''\left(\overline{C}\right) = \frac{2}{\sigma^2}\frac{\lambda}{\rho r} > 0$. Thus, using a Taylor expansion around $C = \overline{C}$, we have

$$f\left(\overline{C} - \varepsilon\right) \approx \frac{1}{2}f''\left(\overline{C}\right)\varepsilon^2 > 0 \tag{124}$$

To show that $f(C) \geq 0$ for all $C \in \left[\underline{C}, \overline{C}\right]$, we proceed by proof of contradiction. Suppose there exists a point $C_0 \in \left(\underline{C}, \overline{C}\right)$ at which $f(C_0) = 0$ — if there are several such points, we the pick the one closest to $\overline{C}$ – which has to be an up-crossing. This implies that there exists a local maximum point $\hat{C} \in \left(C_0, \overline{C}\right)$ at which $f(\hat{C}) > 0$ so $\alpha(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]$ applies,

$$f'(C_0) = 0 \iff P'\left(\hat{C}\right) = \frac{1}{1-\beta\left(\hat{C}\right)}, \tag{125}$$

and $f''\left(\hat{C}\right) < 0$. Evaluating at $C = \hat{C}$, we have

$$f''\left(\hat{C}\right) = -\frac{1}{\left[1-\beta\left(\hat{C}\right)\right]^2}\left[\beta'_{LC}\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right)\right]. \tag{126}$$

To get a contradiction, we need to show $f''\left(\hat{C}\right) > 0 \iff \beta'_{LC}\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right) < 0$. Note that

$$\begin{aligned}
\frac{\sigma^2}{2}\beta'_{LC}\left(\hat{C}\right) &= -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\mu_C\left(C\right) + \frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\pi}{\rho r}\left[P'\left(\hat{C}\right) - 1\right]e^{-\rho r \alpha\left(\hat{C}\right)} - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r} \\
&= -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\left[1 - e^{-\rho r \alpha\left(\hat{C}\right)}\right]\right\} \\
&\quad + \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\pi}{\rho r}\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}e^{-\rho r \alpha\left(\hat{C}\right)} - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r} \\
&= -\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} + \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r}
\end{aligned} \tag{127}$$

Combining, we have

$$\begin{aligned}
\beta'_{LC}\left(\hat{C}\right) + \rho r \beta\left(\hat{C}\right) &= \frac{2}{\sigma^2}\left\{-\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} + \frac{1}{1-\beta\left(\hat{C}\right)}\frac{\rho r}{2}\beta^2\left(\hat{C}\right)\sigma^2 - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\} \\
&= \frac{2}{\sigma^2}\left\{-\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\} + \left[\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)} + 1\right]\rho r \beta\left(\hat{C}\right) \\
&= \frac{2}{\sigma^2}\left\{-\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r}\right\} - \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\} + \frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\rho r \\
&= -\frac{2}{\sigma^2}\left\{\frac{\beta\left(\hat{C}\right)}{1-\beta\left(\hat{C}\right)}\left\{\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r} - \frac{\sigma^2}{2}\rho r\right\} + \frac{\lambda \mathbf{1}_{\{\hat{C}\geq 0\}}}{\rho r}\right\}
\end{aligned} \tag{128}$$

Let us investigate

$$\mu + \left(r - \lambda \mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{r} - \frac{\sigma^2}{2}\rho r = r\left[\frac{\mu}{r} + \left(1 - \frac{\lambda}{r}\mathbf{1}_{\{\hat{C}\geq 0\}}\right)\hat{C} + \frac{\pi}{\rho r^2} - \frac{\sigma^2}{2}\rho\right] \tag{129}$$

Note that the RHS is decreasing in $\hat{C}$, so evaluating $\hat{C} = \underline{C} < 0$, we can show that

$$Y^A + \underline{C} + \frac{\pi}{\rho r^2} = Y^A + \frac{f(\pi) + \frac{\pi}{r}}{\rho r} - Y^A = \frac{w(z(\pi))}{\rho r} > 0, \tag{130}$$

a contradiction.

Consequently, we have $f(C) \geq 0$ and therefore $\alpha_{LC}(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]$ for $C \in \left[\underline{C}, \overline{C}\right]$. The effective HJB under Limited Liability without Commitment can thus be further simplified to

$$r \cdot P(C) = P'(C)\left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right] \tag{131}$$

where $\alpha(C) = \left[P\left(\overline{C}\right) - \overline{C}\right] - \left[P(C) - C\right]$ and β given by (27).

Proof $\lim_{C \to \underline{C}} \mathcal{L}P(C) = 0$**.** What is left is to prove that $\lim_{C \to \underline{C}} \mathcal{L}P(C) = \lim_{C \to \underline{C}} P'(\underline{C})\mu_C(\underline{C}) + P''(\underline{C})\frac{\sigma[1-\beta(\underline{C})]^2}{2} = 0$ under the assumption that $P(\underline{C}) = 0$. First, let us calculate $P'(C)$. Recall the HJB

$$\begin{aligned} rP(C) =& P'(C)\left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right)\right] \\ &+ \pi\left[P\left(\overline{C}\right) - \overline{C} + C - \alpha(C) - P(C)\right] \end{aligned} \tag{132}$$

Rearranging, and taking limits $C \to \underline{C}$, we have

$$\lim_{C \to \underline{C}} P'(C) = \lim_{C \to \underline{C}} \frac{(r + \pi)P(C) - \pi\left[P\left(\overline{C}\right) - \overline{C} + C - \alpha(C)\right]}{\left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right)\right]} \tag{133}$$

First, note that as $C \to \underline{C}$, the denominator converges to $\mu_C(\underline{C}) = 0$ as $\beta(\underline{C}) = \beta^2(\underline{C}) = 1$. Imposing $P(\underline{C}) = 0$, we have

$$\begin{aligned} \lim_{C \to \underline{C}} P'(C) &= \lim_{C \to \underline{C}} \frac{(r + \pi)P(C) - \pi\left[P\left(\overline{C}\right) - \overline{C} + C - \alpha(C)\right]}{\left[\mu + \left(r - \lambda \mathbf{1}_{\{C \geq 0\}}\right)C - \frac{\rho r}{2}\beta(C)\sigma^2 + \pi\left(\frac{1 - e^{-\rho r \alpha(C)}}{\rho r}\right)\right]} \\ &= L'Hopital \\ &= \lim_{C \to \underline{C}} \frac{(r + \pi)P'(C) - \pi\left[1 - \alpha'(C)\right]}{\left[r - \rho r\frac{\sigma^2}{2}\beta'(C) + \pi\alpha'(C)e^{-\rho r \alpha(C)}\right]} \end{aligned} \tag{134}$$

Taking the $P'(C)$ terms from the right to the left hand side, we have

$$\lim_{C\to\underline{C}} P'(C) = \lim_{C\to\underline{C}} \frac{-\pi\left[1-\alpha'(C)\right]}{\left[r-\rho r\frac{\sigma^2}{2}\beta'(C)+\pi\alpha'(C)\,e^{-\rho r\alpha(C)}\right]\left[1-\frac{r+\pi}{\left[r-\rho r\frac{\sigma^2}{2}\beta'(C)+\pi\alpha'(C)e^{-\rho r\alpha(C)}\right]}\right]}$$
$$= \lim_{C\to\underline{C}} \frac{-\pi\left[1-\alpha'(C)\right]}{\left[-\rho r\frac{\sigma^2}{2}\beta'(C)+\pi\left(\alpha'(C)\,e^{-\rho r\alpha(C)}-1\right)\right]} \tag{135}$$

Plugging in for $\beta'(C)$, we have

$$\lim_{C\to\underline{C}} -\pi\left[1-\alpha'(C)\right] = \lim_{C\to\underline{C}} P'(C)\left\{-\rho r\frac{\sigma^2}{2}\beta'(C)+\pi\alpha'(C)\,e^{-\rho r\alpha(C)}-\pi\right\}$$
$$= \lim_{C\to\underline{C}} P'(C)\left\{-\rho r\left[-\frac{\beta(C)}{1-\beta(C)}\mu_C(C)+\frac{\rho r}{2}\beta^2(C)\,\sigma^2+\frac{\pi}{\rho r}\frac{\left[1-\alpha'(C)\right]}{P'(C)}-\frac{\pi}{\rho r}\left[1-\alpha'(C)\,e^{-\rho\right.\right.\right.$$
$$= \lim_{C\to\underline{C}} P'(C)\left\{-\rho r\left[-\frac{\beta(C)}{1-\beta(C)}\mu_C(C)+\frac{\rho r}{2}\beta^2(C)\,\sigma^2-\frac{\lambda\mathbf{1}_{\{C\geq 0\}}}{\rho r}\right]\right\}-\pi\left[1-\alpha'(C)\right] \tag{136}$$

Rearranging, under the assumption that $\underline{C}<0$ we have

$$0 = \lim_{C\to\underline{C}} P'(C)\left\{-\rho r\left[-\frac{\beta(C)}{1-\beta(C)}\mu_C(C)+\frac{\rho r}{2}\beta^2(C)\,\sigma^2\right]\right\}$$
$$= \lim_{C\to\underline{C}} P'(C)\left\{-\rho r\left[-\frac{\beta(C)}{1-\beta(C)}\left[\mu+rC-\frac{\rho r}{2}\beta(C)\,\sigma^2+\pi\left(\frac{1-e^{-\rho r\alpha(C)}}{\rho r}\right)\right]\right]\right\} \tag{137}$$

Thus, the HJB when $C\to\underline{C}<0$ indeed has

$$\lim_{C\to\underline{C}} \mathcal{L}P(C) = 0. \tag{138}$$

Credit line with No-Debt-Forgiveness. From matching the dZ parts, we have (45), which after accounting for $Y(C)$ results in

$$\alpha(C) = \int_C^{\overline{C}} \frac{\beta(x)}{1-\beta(x)}dx \iff \alpha'(C) = -\frac{\beta(C)}{1-\beta(C)}. \tag{139}$$

Note next that the LC constraint (21) is binding.[24] Differentiating (21) w.r.t. C, and plugging into the above restriction, we end up with

$$\alpha'(C) = -\frac{\beta(C)}{1-\beta(C)} = 1 - P'(C) \implies \beta_{NDF}(C) \equiv 1 - \frac{1}{P'(C)}. \tag{140}$$

The NDF contract features sub-optimal risk-sharing $\beta_{NDF}(C) \neq \beta(C)$, leading to a value loss.

Next, note that (47) does not hold anymore, as it was derived from the optimal β, (27). The new credit line drift equation is derived in the Appendix, and yields

$$\mu_T + \mu_I = \mu + (r-\lambda)C^+ - rP(C) < 0. \tag{141}$$

Unlike in the optimal case, the level and not the slope of the value function now matters. Importantly, the previous decomposition still applies: we define $D(C) = \alpha(C) = [P(\overline{C}) - \overline{C}] - [P(C) - C]$ as the balance of the credit line, which then requires an interest rate

$$r_{NDF}(C)D(C) = r\left[\frac{\mu}{r} - \frac{\lambda}{r}C^+ - [P(C) - C^+]\right]. \tag{142}$$

Assuming $r > \lambda$, we can show that the term $[\cdot] > 1$ except at $C = \overline{C}$, and the interest rate is a continuous function of C, with $\lim_{C\to\overline{C}} r_D(C) = r$.

Proof of $C^{survival}$. Recall that LC implies $P(\underline{C}) \geq 0$. As shown in the appendix, $P(\underline{C}) = 0$ implies $\lim_{C\to\underline{C}} P'(\underline{C})\,\mu_C(\underline{C}) + P''(\underline{C})\left[1 - \beta(\underline{C})\right]^2 = 0$. Applied to the HJB (22), this pins down the boundary refinancing payout $\alpha(\underline{C})$ *regardless* of the specific constraint:[25]

$$\alpha(\underline{C}) = P\left(\overline{C}\right) - \left[\overline{C} - \underline{C}\right] = \frac{\mu}{r} - \frac{\lambda}{r}\overline{C} + \underline{C}. \tag{143}$$

[24] If it were not, then we are in the unconstrained case, in which case $\alpha(C) = \alpha_U(C)$, which by the above also fulfills the NDF constraint. But we have shown previously that this unconstrained solution violates LC.

[25] That is, imposing (counter-factually) $P(\underline{C}) = 0$ without restricting α, which results in $\alpha_U(C) = \frac{\log P'(C)}{\rho r}$, would coincide with the two constrained policies at $C = \underline{C}$. XXX

Substituting in for the optimal policies, and using $\alpha\left(\underline{C}\right)$ from above in $\mu_C\left(\underline{C}\right)=0$ while assuming that volatility vanishes, i.e., $\beta\left(\underline{C}\right)=1$, we have

$$\begin{aligned} 0=\mu_C\left(\underline{C}\right) &= \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \pi\frac{1-e^{-\rho r\alpha\left(\underline{C}\right)}}{\rho r} \\ &= \mu + r\underline{C} - \frac{\rho r}{2}\sigma^2 + \frac{\pi}{\rho r}\left(1-e^{-\rho r\left[\frac{\mu}{r}-\frac{\lambda}{r}\overline{C}+\underline{C}\right]}\right). \end{aligned} \tag{144}$$